CS-1104-1 Discrete Mathematics Assignment 1

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DUE: Friday, September 20, 2024

For each problem, explain/justify how you obtained your answer. This will help us determine your understanding of the problem whether or not you got the correct answer. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

Problem 1

Give examples of 4 functions such that the first one is one-one, but not onto, the second one is onto, but not one-one, and the third is neither one-one nor onto and the fourth one is both one-one and onto. In each case prove your claims formally.

The following are the definitions we will be using to solve this problem:

Definition 1 (One-One Function)**.** *[2]*
A function $f : A \longrightarrow B$ is called a one-one function if and only if every element in $Ran(f) \subseteq B$ has a unique pre-image in A.
In other words, $f : A \longrightarrow B$ is a one-one function if and only if $f(x_1) = f(x_2) \implies x_1 = x_2$

Definition 2 (Onto Function)**.** *[2] A function $f : A \longrightarrow B$ is called a onto function if and only if $Ran(f) = B$.*

Definition 3 (One-to-One Function)**.** *[2] A function $f : A \longrightarrow B$ is called one-to-one if and only if it is both one-one and onto. In this case, if $f(a) = b$, we say, $f^{-1}(b) = a$.*

Example 1. *The following is an example of a function which is one-one but not onto:*

$$f : \mathbb{N} \longrightarrow \mathbb{N}$$
$$f(x) = 2x$$

Proof. To prove $f(x)$ is a one-one function.
From **Definition 1**(1), we know that a function is one-one if

$$f(x_1) = f(x_1) \implies x_1 = x_2$$

For $f(x) = 2x$ assume $f(x_1) = f(x_2)$, for any $x_1, x_2 \in \mathbb{N}$

$$2x_1 = 2x_2$$

Divide both sides by 2,

$$x_1 = x_2$$

$\therefore f(x) = 2x$ when $f : \mathbb{N} \longrightarrow \mathbb{N}$ is a one-one function.

Now, we need to show that $f(x)$ when $f : \mathbb{N} \longrightarrow \mathbb{N}$ is not onto. To prove this, we will construct a counterexample.

Let's say $f(x_3) = 3$ for some $x_3 \in \mathbb{N}$
Now, let's try to find the value of x_3:

$$f(x_3) = 2x_3$$
$$\implies 3 = 2x_3$$
$$\implies x_3 = \frac{3}{2}$$

But $x_3 = \frac{3}{2}$ is not a natural number. Therefore, 3 does not have a pre-image in $\mathbb{N}$.

$\therefore f(x) = 2x$ when $f : \mathbb{N} \longrightarrow \mathbb{N}$ is not an onto function. □

Example 2. *The following is an example of a function which is onto but not one-one:*

$$g : \mathbb{R} \longrightarrow \mathbb{R}^{+1}$$
$$g(x) = x^2$$

Proof. First, we show $g(x)$ is an onto function. From **Definition 2**(2), we know that a function is onto if and only if the range of the function is equal to the co-domain. That is, $Ran(g) = \mathbb{R}^+$.

Here, $g(x) = x^2$. The co-domain is $\mathbb{R}^+$ and the range of the function is $[0, \infty)$, since the square of any real number is a non-negative real number.

$\therefore Ran(g) = \mathbb{R}^+$ and $g(x)$ when $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ is an onto function.

Now, we need to prove that $g(x)$ when $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ is not a one-one function.
To show this, we will construct a counterexample.
Let $g(x) = 1$ for some $x \in \mathbb{R}$.

[1] $\mathbb{R}^+$ denotes the set of non-negative real numbers, which includes 0.

$$\begin{aligned} & g(x) = 1 \\ \implies & x^2 = 1 \\ \implies & x = \sqrt{1} \\ \implies & x = 1 \quad or \quad x = -1 \end{aligned}$$

There are two values of x which return the same output for $g(x) = 1$.

$\therefore g(x) = x^2$ when $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ is not a one-one function. □

Example 3. *The following is an example of a function which is neither one-one nor onto:*

$$h : \mathbb{R} \longrightarrow \mathbb{R}$$
$$h(x) = x^2$$

Proof. Firstly, let's prove $h(x)$ is not one-one.

To show this, we will construct a counterexample. Let $h(x) = 2$ for some $x \in \mathbb{R}$.

$$\begin{aligned} & h(x) = 2 \\ \implies & x^2 = 2 \\ \implies & x = \sqrt{2} \\ \implies & x = \sqrt{2} \quad or \quad x = -\sqrt{2} \end{aligned}$$

There are two values of x which return the same output for $h(x) = 2$.

$\therefore h(x) = x^2$ when $h : \mathbb{R} \longrightarrow \mathbb{R}$ is not a one-one function.

Now, let's prove that $h(x)$ is not onto.

From **Definition 2** (2), we know that, a function is considered onto if and only if the range of the function is equal to the co-domain, i.e, $Ran(h) = \mathbb{R}$.

Here, $h(x) = x^2$. The co-domain is $\mathbb{R}$ but the range of the function is $[0, \infty)$ since the square of real numbers is always non-negative.

Therefore, $Ran(h) \neq \mathbb{R}$.

$\therefore h(x) = x^2$ when $h : \mathbb{R} \longrightarrow \mathbb{R}$ is not an onto function. □

Example 4. *The following is an example of a function which is both one-one and onto:*

$$p : \mathbb{N} \longrightarrow \mathbb{N}$$
$$p(x) = x$$

Proof. Firstly, let's show that $p(x) = x$ when $p : \mathbb{N} \longrightarrow \mathbb{N}$ is a one-one function.

By **Definition 1**(1), we know that a function is one-one if and only if for any function $f(x)$, $f(x_1) = f(x_2) \implies x_1 = x_2$ is satisfied. Let's check that for $p(x)$:

$$\begin{aligned} p(x_1) = p(x_2) \ \ for \ \ some \ \ x_1, x_2 \in \mathbb{N} \\ \implies x = x \\ \implies (trivially) \ \ x_1 = x_2 \end{aligned}$$

$\therefore p(x) = x$ when $p : \mathbb{N} \longrightarrow \mathbb{N}$ is a one-one function.

Now, let's show that $p(x)$ is an onto function.

By **Definition 2**(2), we know that, a function is onto if and only if the range of the function is equal to the codomain of the function.

For $p(x)$, the codomain is $\mathbb{N}$ and the range is all $p(x) = x, x \in \mathbb{N}$ which is also $\mathbb{N}$. There can't be a $p(x), x \in \mathbb{N}$ where $p(x) \notin \mathbb{N}$ in this example.

$\therefore p(x) = x$ when $p : \mathbb{N} \longrightarrow \mathbb{N}$ is an onto function. □

Problem 2

Can a relation be both an equivalence relation and a partial order at the same time? Give an example of such a relation and formally argue why it is both.

Yes, a relation can be both an equivalence relation and a partial order at the same time. The relation $R \subseteq A \times A$ defined as $(a, a) \in R$ where $a \in A$ is both an equivalence relation and a partial order relation. To prove this, we will use the following definitions:

Definition 4 (Equivalence Relation)**.** *[2] A relation R on A is called an equivalence relation if and only if R is reflexive, symmetric and transitive.*

Definition 5 (Partial Order Relation)**.** *[2] A relation R on A is called a partial order relation if and only if R is reflexive, anti-symmetric and transitive.*

Definition 6 (Reflexive Relation)**.** *[2] A relation R on A is called reflexive if and only if $\forall a \in A$, we have $(a, a) \in R$.*

Definition 7 (Symmetric Relation)**.** *[2] A relation R on A is called symmetric if and only if $(a, b) \in R \implies (b, a) \in R$.*

Definition 8 (Transitive Relation)**.** *[2] A relation R on A is called transitive if and only if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.*

Definition 9 (Anti-symmetric Relation)**.** *[2] A relation R on A is called anti-symmetric if and only if $(a, b) \in R$ and $(b, a) \in R \implies a = b$.*

The relation $R \subseteq A \times A$ defined as $(a, a) \in R$ where $a \in A$ is both an equivalence relation and a partial order relation.

Proof. First, let's prove $R \subseteq A \times A$ as defined above is an equivalence relation.

To prove equivalence we have to check the following:

1. **Reflexivity:** By Definition 6, $(a, a) \in R \ \forall a \in A$ holds trivially true. Therefore, it is reflexive.

2. **Symmetric:** Let $a \in A$ and $b \in A$. If $(a, b) \in R$, then $(b, a) \in R$ as $a = b$. Therefore, it is symmetric.

3. **Transitive:** Let $a, b, c \in A$. Now, $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$ since by our example, $a = b$ and $b = c$ which $\implies a = c$. Therefore, it is transitive.

This proves $R \subseteq A \times A$ defined as $(a, a) \in R$ where $a \in A$ is an equivalence relation.

Now, to prove it is also a partial order relation, we just need to show it is anti-symmetric:

4. **Anti-symmetry:** [2] Let $a \in A$ and $b \in A$. If $(a, b) \in R$ and $(b, a) \in R$ then, $a = b$, holds true as our relation is defined as $(a, a) \in R \ \forall a \in A$.Therefore, it is anti-symmetric.

This proves $R \subseteq A \times A$ defined as $(a, a) \in R$ where $a \in A$ to be a partial order relation.

$\therefore$ $R \subseteq A \times A$ defined as $(a, a) \in R$ where $a \in A$ is both an equivalence relation and a partial order relation.

□

Problem 3

Prove that the following sets are countable:

1. **Subset of a countably infinite set.**

2. **Union of two countably infinite sets.**

3. **Cartesian product of two countably infinite sets.**

The following definitions will be used to prove the above statements:

Definition 10 (Natural Numbers, $\mathbb{N}$)**.** *[2] $\phi = 0$ is a natural number. If n is a natural number then n^+ (successor of n) is also a natural number ($n^+ = n + 1$). Nothing else is a natural number if it does not satisfy the previous statements.*

Definition 11 (Countably Finite or Finite Set)**.** *[2] A set is called finite if and only if it has a one-to-one correspondence with any any natural number, $n \in \mathbb{N}$.*

Definition 12 (Same Cardinality)**.** *[1] If set A and set B have the same cardinality, then there is a one-to-one correspondence from set A to set B. For a finite set, the cardinality of the set is the number of elements in the set.*

Definition 13 (Countably Infinite)**.** *[1]*
A set A is countably infinite if and only if set A has the same cardinality as $\mathbb{N}$. If set A is countably infinite, then $|A| = |\mathbb{N}|$.

Definition 14 (Countable Set)**.** *[1]*
A set is countable if and only if it is finite or countably infinite.

Part 1

Proof. To show the subset of a countably infinite set is also countable.

Let A countably infinite set, by Definition 13 we know that there exists a bijection $f : \mathbb{N} \longrightarrow A$. Now, assume a new set B, such that $B \subseteq A$.
We aim to prove that B is either finite or countably infinite.

There are 2 possibilities:

1. B is finite. This means it is countable by Definition 11.

2. B is infinite. We will prove that it is countable.

Since $B \subseteq A$, every element of B corresponds to an element in A and as A is countably infinite, we can enumerate the elements as a_n where $n = 0, 1, 2, \ldots$.

Let us define a new mapping $g : \mathbb{N} \longrightarrow B$, where $g(n) = b \ \forall n \in \mathbb{N}$ and $\forall b \in B$. Thus, this maps the elements of $\mathbb{N}$ to those of B. Thus, g creates a bijection between $\mathbb{N}$ and B. By Definition 13, B is countably infinite, which means it is countable.

$\therefore$ A subset of a countably infinite set is countable.

□

Part 2

Proof. To prove the union of two countably infinite sets is countable.

Let A and B be two *countably infinite* sets. We want to prove that the union $A \cup B$ is also countable.
Since A and B are countably infinite, there exist the following bijections:

$$f : A \to \mathbb{N}, \quad \text{and} \quad g : B \to \mathbb{N}.$$

We will construct a new way to enumerate the elements of $A \cup B$. Since A and B are both countably infinite, we can list the elements of A as $a_1, a_2, a_3, \dots$ and the elements of B as $b_1, b_2, b_3, \dots$.

To ensure that every element from $A \cup B$ is listed, we alternate between the elements of A and B, even if they overlap (i.e., $A \cap B \neq \emptyset$).

We can define a sequence that has all the elements of $A \cup B$ as follows:

$$a_1, b_1, a_2, b_2, a_3, b_3, \dots$$

This lists the elements of A and B alternately.

If an element belongs to both A and B (i.e., $A \cap B \neq \emptyset$), we can skip duplicate entries. Since A and B are subsets of $A \cup B$, we are still enumerating all distinct elements of $A \cup B$.

Now, we define a function $h : \mathbb{N} \to A \cup B$ that enumerates all the elements of $A \cup B$. Specifically, for each natural number n, we define $h(n)$ as:

$$h(n) = \begin{cases} a_{(n+1)/2} & \text{if } n \text{ is odd,} \\ b_{n/2} & \text{if } n \text{ is even.} \end{cases}$$

This function ensures that we enumerate all elements from A and B, alternating between them. Since $A \cup B$ is being enumerated by $\mathbb{N}$, it follows that $A \cup B$ is countable.

$\therefore$ The union of two countably infinite sets is countable.

$\square$

Part 3

Proof. To prove the cartesian product of two countably infinite sets is also countable.

Let A and B be two *countably infinite* sets. We want to prove that their cartesian product $A \times B$, defined as the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$, is countable.

Since A and B are countably infinite, there exist the following bijections:

$$f : A \to \mathbb{N}, \quad \text{and} \quad g : B \to \mathbb{N}.$$

We want to enumerate all pairs $(a, b) \in A \times B$. Since A and B are both countably infinite, we can list the elements of A as $a_1, a_2, a_3, \dots$ and the elements of B as $b_1, b_2, b_3, \dots$.

Each element in $A \times B$ is an ordered pair (a_i, b_j), where $a_i \in A$ and $b_j \in B$. The cartesian product consists of all such pairs.

We need to find a way to map the pairs (a_i, b_j) to natural numbers $\mathbb{N}$ in such a way that each pair is assigned a unique natural number.

We can arrange the pairs (a_i, b_j) in a table:

$$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & \dots \\ \hline a_1 & (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \dots \\ a_2 & (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \dots \\ a_3 & (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We can write these pairs diagonally:

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), \dots$$

This creates a one-to-one correspondence between the pairs (a_i, b_j) and the natural numbers $\mathbb{N}$. Therefore, we can define a bijection $h : \mathbb{N} \to A \times B$, which enumerates all the pairs from $A \times B$.

Since, there exists a bijection between $A \times B$ and $\mathbb{N}$, the set $A \times B$ is countable.

$\therefore$ The cartesian product of two countably infinite sets is countable.

$\square$

Problem 4

Prove that the set of infinite binary sequences is uncountable.

Definition 15 (Uncountably Infinite). *[1]*
A set that is NOT countable is uncountable or uncountably infinite.

Proof. WLOG, I will assume the following definitions in context of the question.
Binary Sequence: A binary sequence is a sequence with elements 0 and 1.
Infinite Binary Sequence: An infinite binary sequence is a binary sequence with infinitely many 0's and 1's.

Let B be the set of infinite binary sequences.
Let us prove this by contradiction. Assume the set B is countable, i.e, it can be listed.

Let the elements be:
$b_1 = 01010\dots$
$b_2 = 00011\dots$
$b_3 = 01111\dots$
$b_4 = 01100\dots$
$b_5 = 11010\dots$
and so on, where $b_i \in B$, $i \in \mathbb{N}$.

However, we can construct a new binary sequence, say bin_{new}, such that $bin_{new} \notin B$ where bin_{new} takes the i^{th} element of the i^{th} sequence of B and changes it. Thus, here, $bin_{new} = 11011\ldots$.

This is a contradiction to our assumption that B can be listed as it is countable. So, B is not countable.

$\therefore$ The set of infinite binary sequences is uncountable. $\square$

References

[1] LibreTexts, *Infinite Sets and Cardinality*, Available at: `https://math.libretexts.org/Under_Construction/Stalled_Project_(Not_under_Active_Development)/Additional_Discrete_Topics_(Dean)/Infinite_Sets_and_Cardinality#:~:text=An%20infinite%20set%20that%20can,with%20N%20is%20uncountably%20infinite` (Accessed on: September 19, 2024).

[2] Anwesha Ghosh, *Personal Notes on Discrete Mathematics*, Unpublished, 2024.