CS-1104-1 Discrete Mathematics Assignment 2

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For each problem, explain/justify how you obtained your answer. This will help us determine your understanding of the problem whether or not you got the correct answer. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide.

Problem 1

Solve the following by induction. Mention what type of induction you are using.

1. **Find the sum of the n^{th} term of a general geometric series.**
2. **State and prove the Binomial Theorem for $(a+b)^n$ for any natural number n.**

Solution 1.1

Using Weak Induction

Proof. We have to prove the sum of the n^{th} term of a general geometric series is as follows:

$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for $r \neq 0$

Base Case $(n = 1)$

For n = 1, the sum is the first term of the series, $S_1 = a$.

We can plug this in the formula:

$S_1 = \frac{a(1-r^1)}{1-r} = \frac{a(1-r)}{1-r} = a$ for $r \neq 0$

$\therefore$ The Base Case holds.

Induction Hypothesis $(n = k)$

Assume that that formula holds for for $n = k$. This gives:

$S_k = a + ar + ar^2 + \cdots + ar^{k-1} = \frac{a(1-r^k)}{1-r}$ for $r \neq 0$

Induction Step (Now to prove for $n = k + 1$)

We need to show: $a + ar + ar^2 + \cdots + ar^k = \frac{a(1-r^{k+1})}{1-r}$

$$\begin{aligned}
&S_{k+1} = a + ar + ar^2 + \cdots + ar^k \\
\implies &S_{k+1} = a + ar + ar^2 + \cdots + ar^{k-1} + ar^k \\
\implies &S_{k+1} = S_k + ar^k \\
\implies &S_{k+1} = \frac{a(1-r^k)}{1-r} + ar^k \\
\implies &S_{k+1} = \frac{a(1-r^k) + ar^k(1-r)}{1-r} \\
\implies &S_{k+1} = \frac{a(1-r^k+r^k-r^{k+1})}{1-r} \\
\implies &S_{k+1} = \frac{a(1-r^{k+1})}{1-r}
\end{aligned}$$

$\therefore S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for $r \neq 0$ is proved. □

Solution 1.2

Using Weak Induction

Binomial Theorem Statement
For any natural number n, the expansion of $(a+b)^n$ is given by:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. **Base Case** ($n = 1$)
For $n = 1$,

Left Hand Side (LHS):

$$(a+b)^1 = a + b$$

Right Hand Side (RHS):

$$\binom{1}{0} a^1 b^0 + \binom{1}{1} a^0 b^0 = a + b$$

Since, LHS = RHS,
$\therefore$ Base case holds.

Induction Hypothesis $(n = k)$
Assume that the statement holds true for $n = k$, i.e.,

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i$$

$$
\begin{aligned}
& (a+b)^{k+1} = (a+b)(a+b)^k \\
\implies & (a+b)^{k+1} = (a+b) \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i \\
\implies & (a+b)^{k+1} = a \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i + b \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i \\
\implies & (a+b)^{k+1} = \sum_{i=0}^{k} \binom{k}{i} a^{k+1-i} b^i + \sum_{i=1}^{k+1} \binom{k}{i-1} a^{k+1-i} b^i \\
& \text{(By Pascal's identity: } \binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}\text{)} \\
\implies & (a+b)^{k+1} = a^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} a^{k+1-i} b^i + b^{k+1} \\
\implies & (a+b)^{k+1} = \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k+1-i} b^i
\end{aligned}
$$

Thus, the Binomial Theorem holds for $n = k + 1$.

$\therefore$ The Binomial Theorem holds for all natural numbers n:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$\square$

Problem 2

Prove the following by the Well-Ordering Principle:

1. **21 divides $4^{n+1} + 5^{2n-1}$, for any positive integer n.**
2. **Every positive integer greater than 1 can be factored as a product of primes.**

Solution 2.1

Well-Ordering Principle: Every non-empty set of non-negative integers has a least element.

Proof. **Define the set**
Let's define a set S that contains all positive integers n for which 21 does not divide $4^{n+1}+5^{2n-1}$. That is,

$$S = \{n \in \mathbb{N} : 21 \text{ does not divide } 4^{n+1} + 5^{2n-1}\}.$$

Assume S is non-empty Assume, for contradiction, that S is non-empty. By the Well-Ordering Principle, S must contain a least element, say n_0. This means that 21 does not divide $4^{n_0+1} + 5^{2n_0-1}$, but for all integers smaller than n_0, 21 divides $4^{n+1} + 5^{2n-1}$.
Base case check
We first check the base case for $n = 1$:

$$4^{1+1} + 5^{2(1)-1} = 4^2 + 5^1 = 16 + 5 = 21.$$

Since 21 divides 21, the statement holds for $n = 1$, i.e., $n_0 \neq 1$.
Contradiction
Now, let's consider n_0, the smallest element in S. Since we have assumed that 21 does not divide $4^{n_0+1} + 5^{2n_0-1}$, this contradicts the fact that for all smaller values of n, 21 divides $4^{n+1} + 5^{2n-1}$. Thus, if 21 divides the expression for all smaller values, it must also divide it for n_0. Therefore, our assumption that S is non-empty leads to a contradiction.
Conclusion
Since assuming the set S is non-empty leads to a contradiction, it must be the case that S is empty. Therefore, for all $n \in \mathbb{N}$, 21 divides $4^{n+1} + 5^{2n-1}$. Thus, the statement holds for all positive integers n. □

Solution 2.2

Every positive integer greater than 1 can be factored as a product of primes.

Proof. **Define the set:** Let $S = \{n \in \mathbb{N} : n > 1 \text{ and } n \text{ cannot be factored as a product of primes}\}$.

Assumption: Assume $S \neq \emptyset$.
By the Well-Ordering Principle, S has a least element, say n_0.

Case 1: If n_0 is prime, then it is already a product of primes, contradicting the definition of S.

Case 2: If n_0 is composite, then $n_0 = a \times b$ where $1 < a, b < n_0$. Since a and b are smaller than n_0, they can be factored into primes by the minimality of n_0. Hence, n_0 is a product of primes, contradicting the assumption that $n_0 \in S$.

Conclusion: Therefore, S must be empty, meaning every integer greater than 1 can be factored as a product of primes. □

Problem 3

Given a positive real number r, suppose one proves by induction that $r^n = 1$ for all natural numbers n. The proof goes as follows. The base case for $n = 0$ is established first. Then assuming that the result holds for all $a \leq n$, the proof uses strong induction to show the following: $r^{n+1} = r^n \cdot r^n / r^{n-1} = 1 \cdot 1/1 = 1$. Find the fallacy in the proof with proper justification.

Solution 3

1. **Incorrect Use of Strong Induction:** The induction hypothesis assumes that $r^k = 1$ for all integers $k \leq n$. The proof then proceeds to show $r^{n+1} = 1$ by using the expression:

$$r^{n+1} = \frac{r^n \cdot r^n}{r^{n-1}} = \frac{1 \cdot 1}{1} = 1.$$

The equation $r^{n+1} = \frac{r^n \cdot r^n}{r^{n-1}}$ is not valid unless $r = 1$. The use of $r^n \cdot r^n$ suggests multiplying powers of r, but this doesn't directly follow from the induction hypothesis unless $r = 1$. For arbitrary r, this step doesn't hold.

2. **Induction Step Fallacy:** The critical error occurs because the proof does not address the possibility that $r \neq 1$. The induction hypothesis assumes $r^n = 1$, but the step:

$$r^{n+1} = r^n \cdot r = r$$

shows that, unless $r = 1$, r^{n+1} is not equal to 1. Thus, the hypothesis that $r^n = 1$ for all n breaks down unless $r = 1$.

$\therefore$ The fallacy in the proof lies in the assumption that the strong induction hypothesis automatically implies $r^{n+1} = 1$, regardless of r. In fact, $r^n = 1$ for all n is only true if $r = 1$. The induction step fails to hold for any $r \neq 1$. Therefore, the conclusion that $r^n = 1$ for all n is false unless $r = 1$.