Complex Analysis Assignment 1

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Problem 1

Consider the function (defined using power series in class)

$$\sin : \mathbb{C} \to \mathbb{C}$$

$$z \mapsto \sin(z)$$

Is it a bounded function? If yes, give a proof. If no, give a couterexample with explanation.

Solution

The function $\sin(z)$ is not a bounded function over the complex plane.

Proof. Let us look at the power series definition of sin(z):

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Here, $z = x + iy$ where $x, y \in \mathbb{R}$. Now, let us expand this.

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

Substituting $z = x + iy$:

$$\sin(x + iy) = (x + iy) - \frac{(x + iy)^3}{3!} + \frac{(x + iy)^5}{5!} - \dots$$

If $y = 0 \implies \sin(x) \in \mathbb{R}^2$. But we want sin(z) to be in $\mathbb{C}$. So, we consider the case where

$x = 0 \implies \sin(iy)$.

$$\begin{aligned}\sin(iy) &= iy - \frac{(iy)^3}{3!} + \frac{(iy)^5}{5!} - \frac{(iy)^7}{7!} + \dots \\ &= iy + \frac{i^3y^3}{3!} - \frac{i^5y^5}{5!} + \frac{i^7y^7}{7!} - \dots \\ &= iy + \frac{iy^3}{3!} + \frac{iy^5}{5!} + \frac{iy^7}{7!} + \dots \\ &= i\left(y + \frac{y^3}{3!} + \frac{y^5}{5!} + \frac{y^7}{7!} + \dots\right) \\ &= i\sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!}\end{aligned}$$

Let us denote the series $\sum_{n=0}^{\infty} \frac{y^{2n+1}}{(2n+1)!}$ as $S(y)$.

Now, $\forall y \geq 0 (\in \mathbb{R}), S(y) \geq y$. As $y \to \infty$, $S(y) \to \infty$. Therefore, $\sin(iy) = iS(y)$ also becomes unbounded as $y \to \infty$.

□

Problem 2

Observe that

$$\lim_{n\to\infty} e^{2\pi i n} = 1.$$

Let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ **such that**

$$f(z) := e^{1/z}.$$

Show that for any complex number c **there exists a sequence** $\{c_n\}_{n=1}^{\infty}$ **of complex numbers such that**

$$\lim_{n\to\infty} c_n = 0, \qquad \lim_{n\to\infty} f(c_n) = c.$$

Solution

From our observation, we know that $\lim_{n\to\infty} e^{2\pi i n} = 1$. This tells us that the exponential function is periodic. So, $\lim_{n\to\infty} e^{w+2\pi i n} = e^w$.

Claim: For every $c \in \mathbb{C}$ *there exists a sequence* $\{c_n\}_{n=1}^{\infty}$ *such that* $\lim_{c_n\to\infty} f(c_n) = c$.

Proof. We can divide this proof into two cases, based on the value of c.

Case 1: c = 0
Let $c_n = \frac{1}{-n}$ where $n \in \mathbb{R}^+$.
Then, as $n \to \infty$, $c_n \to 0$ and $f(c_n) = e^{1/c_n} = e^{-n}$. We know $\lim_{n\to\infty} f(c_n) = \lim_{n\to\infty} e^{-n} = 0$.

Case 2: $c \neq 0$

Let us construct a sequence $\{c_n\}_{n=1}^{\infty}$ such that it satisfies the conditions of the problem statement.
Choose $w \in \mathbb{C}$ such that $e^w = c$. This is possible since the exponential function is surjective onto $\mathbb{C} \setminus \{0\}$. Now, for $n \in \mathbb{N}$, define the sequence:

$$c_n = \frac{1}{w + 2\pi i n}$$

Also, as $n \to \infty$, $w + 2\pi i n \to \infty$ and hence $c_n \to 0$.
Now, we plug this in $f(c_n)$:

$$f(c_n) = e^{1/c_n} = e^{w+2\pi i n} = e^w \cdot e^{2\pi i n} = c \cdot 1 = c$$

So, $\lim_{n\to\infty} f(c_n) = c$.

$\therefore$ We have constructed a sequence $\{c_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} c_n = 0$ and $\lim_{n\to\infty} f(c_n) = c$ for any $c \in \mathbb{C}$. □

Problem 3

Let $U \subseteq \mathbb{C}$ be an open subset and $f : U \to \mathbb{C}$ be complex differentiable everywhere. Show the following:

(1) If f is real-valued then f is constant.

(2) If $Re(f)$ is constant then f is constant.

(3) If $|f|$ is constant then f is constant.

Solution (1)

Proof. We know that,

$$f : U \to \mathbb{C}$$
$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y) = Re(f)$ and $v(x, y) = Im(f)$.
Since f is real-valued, $v(x, y) = 0 \forall (x, y) \in U$.
Now, since f is complex differentiable everywhere in U, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Substituting $v(x, y) = 0$:

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

This implies that u is constant throughout U. Therefore, $f(z) = u(x, y) + i \cdot 0 = u(x, y)$ is constant. □

Solution (2)

Proof. Given that $Re(f)$ is constant, let $Re(f) = c$ for some constant $c \in \mathbb{R}$. Thus, we can write:

$$f(z) = c + iv(x, y)$$

where $v(x, y) = Im(f)$.

Since f is complex differentiable everywhere in U, it satisfies the Cauchy-Riemann equations. Now, substituting $u(x, y) = c$:

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

This implies that:

$$\frac{\partial v}{\partial y} = 0 \quad \text{and} \quad -\frac{\partial v}{\partial x} = 0$$

Thus, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$, which means v is constant throughout U. Therefore, $f(z) = c + i \cdot v(x, y)$ is constant.

□

Solution (3)

Proof. Given that $|f|$ is constant, let $|f| = r$ for some constant $r \geq 0$. Thus, we can write:

$$|f(z)| = \sqrt{u(x, y)^2 + v(x, y)^2} = r$$

Squaring both sides, we get:

$$u(x, y)^2 + v(x, y)^2 = r^2$$

Differentiating both sides with respect to x and y:

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad (1)$$

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \quad (2)$$

Since f is complex differentiable everywhere in U, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (3)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (4)$$

Substituting equations (3) and (4) into (1) and (2): From (1):

$$2u\frac{\partial u}{\partial x} + 2v\left(-\frac{\partial u}{\partial y}\right) = 0$$

From (2):

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial u}{\partial x} = 0$$

This can be written in matrix form:

$$\begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The determinant of the coefficient matrix is:

$$(u^2 + v^2) = r^2$$

Since $r \neq 0$, the determinant is non-zero, which implies that the only solution to the system is

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0$$

This means that u is constant throughout U. Substituting back into the equation $u(x, y)^2 + v(x, y)^2 = r^2$, we find that v must also be constant. Therefore, $f(z) = u(x, y) + iv(x, y)$ is constant. □

Problem 4

Suppose $I : \mathbb{C}^2 \to \mathbb{R}^4$ and $J : \mathbb{R}^4 \to \mathbb{C}^2$ are such that

$$I(a + ib, c + id) = (a, b, c, d), \quad J(a, b, c, d) = (a + ib, c + id).$$

Answer the following questions with justifications:

(1) Suppose V is a 1-dimensional vector subspace of the vector space $\mathbb{C}^2$ over the field $\mathbb{C}$. Is $I(V)$ a 2-dimensional vector subspace of $\mathbb{R}^4$ over the field $\mathbb{R}$?

(2) Suppose W is a 2-dimensional vector subspace of the vector space $\mathbb{R}^4$ over the field $\mathbb{R}$. Is $J(W)$ a 1-dimensional vector subspace of $\mathbb{C}^2$ over the field $\mathbb{C}$?

Solution (1)

Proof. Let us first show that I is a real linear transformation.
We know that for any $z_1, z_2 \in \mathbb{C}^2$ and $\alpha, \beta \in \mathbb{R}$:

$$I(\alpha z_1 + \beta z_2) = \alpha I(z_1) + \beta I(z_2)$$

This shows that I is a real linear transformation.
Now, let V be a 1-dimensional vector subspace of $\mathbb{C}^2$ over $\mathbb{C}$. This means that there exists a non-zero vector $v \in \mathbb{C}^2$ such that:

$$V = \{\lambda v : \lambda \in \mathbb{C}\}$$

Let $v = (z_1, z_2)$ where $z_1, z_2 \in \mathbb{C}$ and $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ where $a_1, b_1, a_2, b_2 \in \mathbb{R}$.
Then,

$$I(v) = I(z_1, z_2) = (a_1, b_1, a_2, b_2) \in \mathbb{R}^4$$

Now, consider any vector $w \in V$.

$$w = \lambda v = \lambda(z_1, z_2) = (\lambda z_1, \lambda z_2)$$

where $\lambda = x + iy$ for some $x, y \in \mathbb{R}$.
Then,

$$I(w) = I(\lambda z_1, \lambda z_2) = (xa_1 - yb_1, xb_1 + ya_1, xa_2 - yb_2, xb_2 + ya_2)$$

The set $I(V)$ is spanned by the vectors:

$$(a_1, b_1, a_2, b_2) \quad \text{and} \quad (-b_1, a_1, -b_2, a_2)$$

Since these two vectors are linearly independent (as $v \neq 0$), $I(V)$ is a 2-dimensional vector subspace of $\mathbb{R}^4$ over the field $\mathbb{R}$.

□

Solution (2)

Proof. No, $J(W)$ is not necessarily a 1-dimensional vector subspace of $\mathbb{C}^2$ over the field $\mathbb{C}$.
Let us construct a counterexample.
Consider the 2-dimensional vector subspace W of $\mathbb{R}^4$ spanned by the vectors:

$$(1, 0, 0, 0) \quad \text{and} \quad (0, 0, 1, 0)$$

Then,

$$J(1, 0, 0, 0) = (1 + i0, 0 + i0) = (1, 0)$$
$$J(0, 0, 1, 0) = (0 + i0, 1 + i0) = (0, 1)$$

The set $J(W)$ contains vectors $(x, 0)$ and $(0, y)$ for $x, y \in \mathbb{R}$. So, obviously, they don't contain any complex scalar multiples of each other. This means that $J(W)$ is not closed under multiplication by complex scalars. Therefore, $J(W)$ is not a vector subspace of $\mathbb{C}^2$ over the field $\mathbb{C}$, let alone being 1-dimensional. □

Problem 5

Answer the following questions with justifications:

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = |x|^2$. Is f real differentiable everywhere? Now consider the extended function $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = |z|^2$. Is f complex differentiable everywhere?

(2) Let $g : \mathbb{R} \to \mathbb{R}$ be such that

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Is g real differentiable everywhere? Now consider the extended function $g : \mathbb{C} \to \mathbb{C}$ defined by

$$g(z) = \begin{cases} z^2 \sin\left(\frac{1}{z}\right), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Is g complex differentiable everywhere?

Solution (1)

Proof. The function $f(x) = |x|^2$ is real differentiable everywhere on $\mathbb{R}$. The derivative is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2xh + h^2}{h} = 2x$$

Thus, $f'(x) = 2x$ exists for all $x \in \mathbb{R}$.
Now, consider the extended function $f(z) = |z|^2$ for $z \in \mathbb{C}$. We can write:

$$f(z) = x^2 + y^2$$

To check if f is complex differentiable, we use the Cauchy-Riemann equations. Let $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. The Cauchy-Riemann equations state that:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Calculating the partial derivatives:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

Substituting these into the Cauchy-Riemann equations:

$$2x = 0 \quad \text{and} \quad 2y = 0$$

These equations hold only at the point $(0, 0)$. Therefore, $f(z) = |z|^2$ is not complex differentiable everywhere in $\mathbb{C}$, only at $z = 0$. □

Solution (2)

Proof. Real differentiability: For $x \neq 0$,

$$g'(x) = \frac{d}{dx}\big(x^2 \sin(1/x)\big) = 2x\sin(1/x) + x^2\cos(1/x)\cdot(-1/x^2) = 2x\sin(1/x) - \cos(1/x).$$

At $x = 0$, we check the difference quotient:

$$\lim_{h\to 0} \frac{g(h) - g(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = \lim_{h\to 0} h\sin(1/h).$$

Since $|h\sin(1/h)| \leq |h| \to 0$, the limit exists and equals 0. Thus

$$g'(0) = 0.$$

Therefore g is differentiable at every real x.

Complex differentiability:

At $z = 0$ we check the derivative:

$$\lim_{z\to 0} \frac{g(z) - g(0)}{z} = \lim_{z\to 0} z\sin\left(\tfrac{1}{z}\right).$$

Take $z_n = \frac{1}{n} \to 0$. Then

$$z_n \sin\left(\tfrac{1}{z_n}\right) = \frac{1}{n}\sin(n).$$

Since $|\sin(n)| \leq 1$, we have

$$\left|\frac{1}{n}\sin(n)\right| \leq \frac{1}{n} \to 0.$$

So the limit along the real axis is 0.

Take $w_n = \frac{i}{n} \to 0$. Then

$$\frac{1}{w_n} = -in, \qquad \sin\left(\tfrac{1}{w_n}\right) = \sin(-in).$$

Using $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$,

$$\sin(-in) = \frac{e^{i(-in)} - e^{-i(-in)}}{2i} = \frac{e^n - e^{-n}}{2i}.$$

Hence

$$w_n \sin\left(\tfrac{1}{w_n}\right) = \frac{i}{n}\cdot\frac{e^n - e^{-n}}{2i} = \frac{e^n - e^{-n}}{2n}.$$

As $n \to \infty$, this grows without bound.

Along the real axis, the limit is 0, while along the imaginary axis, the difference quotients are unbounded. Therefore

$$\lim_{z2\to 0} z^2 \sin\left(\tfrac{1}{z}\right)$$

does not exist as a finite complex number. Hence g is not complex differentiable at 0. □

Problem 6

Suppose $U \subseteq \mathbb{C}$ is an open set, and let $h : U \to \mathbb{C}$ be a function which is complex differentiable everywhere. Show that the function $H : U \to \mathbb{C}$ defined by

$$H(z) := \overline{h(\overline{z})}$$

is also complex differentiable everywhere.

Solution

Proof. Let $h(z) = u(x, y) + iv(x, y)$ where $z = x + iy$ and $u, v : U \to \mathbb{R}$. Since h is complex differentiable everywhere in U, it satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Now, consider the function $H(z) = \overline{h(\overline{z})}$. Substituting $z = x + iy$:

$$H(z) = \overline{h(x - iy)} = \overline{u(x, -y) + iv(x, -y)} = u(x, -y) - iv(x, -y)$$

Let $U(x, y) = u(x, -y)$ and $V(x, y) = -v(x, -y)$. Thus, we can write:

$$H(z) = U(x, y) + iV(x, y)$$

To show that H is complex differentiable, we need to verify that U and V satisfy the Cauchy-Riemann equations:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

Calculating the partial derivatives:

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x}(x, -y), \quad \frac{\partial U}{\partial y} = -\frac{\partial u}{\partial y}(x, -y)$$

and

$$\frac{\partial V}{\partial x} = -\frac{\partial v}{\partial x}(x, -y), \quad \frac{\partial V}{\partial y} = -\left(-\frac{\partial v}{\partial y}(x, -y)\right) = \frac{\partial v}{\partial y}(x, -y)$$

Now, substituting these into the Cauchy-Riemann equations:

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x}(x, -y) = \frac{\partial v}{\partial y}(x, -y) = \frac{\partial V}{\partial y}$$

and

$$\frac{\partial U}{\partial y} = -\frac{\partial u}{\partial y}(x, -y) = -\left(-\frac{\partial v}{\partial x}(x, -y)\right) = -\frac{\partial V}{\partial x}$$

Thus, U and V satisfy the Cauchy-Riemann equations, which implies that H is complex differentiable everywhere in U. $\square$