Department of Mathematics
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MAT-3018: Complex Analysis

Assignment 2

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Problem 1

Consider the function

$$u : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^3 - 3xy^2$$

1. Find a harmonic conjugate using the first method we saw in class, via integration.
2. Now find a harmonic conjugate using the method from Ahlfors (see pg. 27). [Hint: Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a harmonic function which is a polynomial (in two variables x, y). Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) := 2u\Big(\frac{z}{2}, \frac{z}{2i}\Big) - u(0, 0)$$

is holomorphic and its real part is u.

Solution 1:

Part 1

Let us prove that u is a harmonic function.
We have:

$$u_x = 3x^2 - 3y^2, u_y = -6xy$$

$$u_{xx} = 6x, u_{yy} = -6x$$

Thus, $u_{xx} + u_{yy} = 0$. Therefore, u is a harmonic function.

Let it's conjugate be v. The method to find the harmonic conjugate:

$$\begin{aligned} v(x, y) &= \int_0^y u_x(x, t)dt + \int_0^x u_y(s, 0)ds \\ &= \int_0^y (3x^2 - 3t^2)dt - \int_0^x 6s \cdot 0ds \\ &= 3x^2y - y^3 \end{aligned}$$

Therefore, the harmonic conjugate of $v(x, y) = 3x^2y - y^3$.

Part 2

Proof: We know u is a harmonic polynomial in two variables x, y, so we can see that $f(z)$ is a polynomial too. This implies that it is holomorphic.

Now, let us compute out $f(z)$.

$$\begin{aligned} f(z) &= 2u(\frac{z}{2}, \frac{z}{2i}) - u(0, 0) \\ &= 2((\frac{z}{2})^3 - 3\frac{z}{2}(\frac{z}{2i})^2) - 0 \\ &= \frac{z^3}{4} + \frac{3z^3}{4} \\ &= z^3 \\ &= (x + iy)^3 \\ &= (x^3 - 3xy^2) + i(-y^3 + 3x^2y) \end{aligned}$$

Since $x, y \in \mathbb{R}$, we see that the u defined in the problme is indeed the real part of f.

We know from the Alhfors method that the harmonic conjugate is the imaginary part of f. Therefore the harmonic conjugate is $y^3 + 3x^2y$. This matches up with the first part of the solution where we used the integration method.

□

Problem 2

Let N be the last two digits of your Ashoka ID, define a function

$$\log_N : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad \log_N(z) := \log|r| + i\theta,$$

where $z = r\exp(i\theta)$ and $-\pi + N \leq \theta < \pi + N$.
Is it true that for any two non-zero complex numbers z, w we have

$$\log(zw) = \log(z) + \log(w)?$$

Solution 2:

Consider $N = 55$.
We define the function: $\log_{55}(z) = \log|r| + i\theta, -\pi + 55 \leq \theta < \pi + 55$, where $z = r\exp(i\theta)$.

We want to know if for any two non-zero complex numbers z, w we have

$$\log_{55}(zw) = \log_{55}(z) + \log_{55}(w)$$

. Let $z = w = 1$. Then, $zw = 1$.

Now, $\log_{55}(zw) = \log_{55}(1) = \log|1| + i\theta$. The real logarithm of 1 is 0. The argument θ is 0. But 0 is not in the interval $[-\pi + 55, \pi + 55)$.

We take $\theta = 0 + 2\pi k$. The interval $[-\pi + 55, \pi + 55)$ is equivalent to $[51.8584, 58.1416)$. Take $k = 9$. This implies $\theta = 18\pi = 56.5486$, which belongs in the interval. Thus, $\log_{55}(zw) = 0 + i(18\pi) = i(18\pi)$.

Similarly, $\log_{55}(z) = \log_{55}(1) = i(18\pi)$ and $\log_{55}(w) = \log_{55}(1) = i(18\pi)$. But, $\log_{55}(z) + \log_{55}(w) = i(18\pi) + i(18\pi) = i(36\pi)$.

Thus, $\log_{55}(zw) \neq \log_{55}(z) + \log_{55}(w)$ for $z = w = 1$. Hence, the statement is false.

Problem 3

Let $\Sigma = \mathbb{C} \cup \{\infty\}$.

(1) Let z_1, z_2, z_3, z_4 and w_1, w_2, w_3, w_4 be points on Σ. Can we always find a Möbius transformation T such that $T(z_i) = w_i$, for $1 \leq i \leq 4$? Give proof or a concrete non-example.

(2) Let z_1, z_2, z_3 and w_1, w_2, w_3 be points on Σ. Can we always find a Möbius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 3$? Give proof or a concrete non-example.

Solution 3:

I will first do part (2) and then part (1).

Part (2):

Proof: Let z_1, z_2, z_3 and w_1, w_2, w_3 be points on Σ. We want to find a Mobius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 3$.

Let S be a Mobius transformation such that $S(z_1) = \infty, S(z_2) = 0, S(z_3) = 1$. Such a Mobius transformation exists and is given by:

$$S(z) = \frac{(z - z_2)(z_3 - z_1)}{(z - z_1)(z_3 - z_2)}$$

To check that this is indeed a Mobius transformation, we can write it as:

$$S(z) = \frac{z(z_3 - z_1) - (z_2z_3 - z_1z_2)}{z(z_3 - z_2) - (z_1z_3 - z_1z_2)}$$

This is of the form $\frac{az+b}{cz+d}$, where $a = z_3 - z_1, b = -(z_2z_3 - z_1z_2), c = z_3 - z_2, d = -(z_1z_3 - z_1z_2)$.

The determinant of this transformation is $ad-bc = (z_3-z_1)(-(z_1z_3-z_1z_2))-(-(z_2z_3-z_1z_2))(z_3-z_2) = (z_3-z_1)(z_3-z_2)(z_2-z_1)$. This is non-zero as z_1, z_2, z_3 are distinct. Therefore, S is a Mobius transformation.

Similarly, let's construct a Mobius transformation R such that $R(\infty) = w_1, R(0) = w_2, R(1) = w_3$. It is given by inverse of the above transformation.

Note

The inverse of a 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\frac{1}{(z_3-z_1)(z_3-z_2)(z_2-z_1)}\begin{pmatrix} -z_1(z_3-z_2) & z_2(z_3-z_1) \\ (z_2-z_3) & (z_3-z_1) \end{pmatrix}$$

Since, z_1, z_2, z_3 are arbitrary distinct points, put them as w_1, w_2, w_3 respectively.
Thus, $R(z) = \frac{z(\frac{-z_1}{(z_3-z_1)(z_2-z_1)})+\frac{z_2}{(z_3-z_2)(z_2-z_1)}}{z(\frac{-1}{(z_3-z_1)(z_2-z_1)})+\frac{1}{(z_3-z_2)(z_2-z_1)}}$. We know inverse of a Mobius transformation is also a Mobius transformation. So R is a valid Mobius transformation.

Now, consider the composition of the two Mobius transformations $T = R \circ S$.
Then, $T(z_1) = R(S(z_1)) = R(\infty) = w_1$. Similarly, $T(z_2) = w_2$ and $T(z_3) = w_3$. Thus, we have found a Mobius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 3$.

Further, the Mobius transformation is *unique*. Assume there exists another Mobius transformation T' such that $T'(z_i) = w_i$ for $1 \leq i \leq 3$.

Consider the Mobius transformation $U = T' \circ T^{-1}$. Then, $U(w_i) = T'(T^{-1}(w_i)) = T'(z_i) = w_i$ for $1 \leq i \leq 3$. This fixes three points. Now, a Mobius transformation that fixes three points is the identity transformation. Let's do a quick proof of that.

Let $U(z) = \frac{az+b}{cz+d}$ be a Mobius transformation that fixes three points w_1, w_2, w_3. A fixed point is a point that remains unchanged by the transformation. This implies:

$$\frac{az+b}{cz+d} = z$$

$$\Rightarrow cz^2 + (d-a)z - b = 0$$

This is a quadratic equation. It can have at most two distinct roots. But we have three distinct fixed points. Thus, the only possibility is that $c = 0, d - a = 0, -b = 0$. This implies $a = d$ and $b = 0$. Thus, $U(z) = \frac{az}{d} = z$. This is the identity transformation.

Using this result, we have $T' \circ T^{-1}$ is the identity transformation. This implies $T' = T$. Thus, the Mobius transformation is unique.
Hence, we have proved that for any three distinct points z_1, z_2, z_3 and w_1, w_2, w_3 on Σ, there exists a unique Mobius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 3$.
□

Note

Here I have assumed z_i's and w_i's are distinct. But even if they are equal, things will be fine.
Consider, WLOG $z_1 = z_2$, and $w_1 = w_2$. Then you can take any $z_4 \neq z_1$, $w_4 \neq w_1$.
From the above where we assumed it to be distinct, we can have a Mobius transformation that maps $(z_1, z_3, z_4) \to (w_1, w_3, w_4)$.
$z_1 \to w_1$, $z_2 \to w_2$, $z_3 \to w_3$.
Similarly, this can be done for 3 of the elements being equal.

Part (1):

Proof:

Let z_1, z_2, z_3, z_4 and w_1, w_2, w_3, w_4 be points on Σ. We want to find a Mobius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 4$.

Put $z_1 = w_1, z_2 = w_2, z_3 = w_3$. From part (2), we know there exists a unique Mobius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 3$ and this transformation is the identity transformation.

Now, consider z_4 and w_4. If $z_4 = w_4$, then $T(z_4) = w_4$. But if $z_4 \neq w_4$, then $T(z_4) \neq w_4$ as T is the identity transformation. Therefore, there does not always exist a Mobius transformation T such that $T(z_i) = w_i$ for $1 \leq i \leq 4$ for any arbitrary points z_1, z_2, z_3, z_4 and w_1, w_2, w_3, w_4 on Σ.

□

Problem 4

Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(0) = 0, \quad f(x) = \exp\left(-\frac{1}{x^2}\right) \quad \text{for } x \neq 0$$

is smooth (i.e. infinitely differentiable). Consider the Taylor series centered at 0,

$$T_{f,0}(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Show that $T_{f,0}$ is identically the 0 function on all of $\mathbb{R}$ and so it cannot be the function f.
Hint: Show via induction that for $x > 0$ and $n \geq 1$, $f^{(n)}(x)$ is of the form $p_n(x^{-1})f(x)$ for some polynomial $p_n(Y)$ in the variable Y.

Solution 4:

Proof:
To show that f is infinitely differentiable, we will prove that $f^{(n)}(x)$ exists and is continuous for all $n \geq 0$.

Claim: For $x \neq 0$ and $n \geq 1$, $f^{(n)}(x)$ is of the form $p_n(x^{-1})f(x)$ for some polynomial $p_n(Y)$ in the variable Y.
We will prove this by induction on n.

Base Case: For $n = 1$, we have:

$$f'(x) = \frac{d}{dx}\left(e^{-\frac{1}{x^2}}\right) = e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} = p_1(x^{-1})f(x)$$

where $p_1(Y) = 2Y^3$. Thus, the base case holds.

Induction Hypothesis: Assume that for some $k \geq 1$, $f^{(k)}(x) = p_k(x^{-1})f(x)$ for some polynomial $p_k(Y)$.

Induction Step: We need to show that $f^{(k+1)}(x) = p_{k+1}(x^{-1})f(x)$ for some polynomial $p_{k+1}(Y)$.
We have:

$$f^{(k+1)}(x) = \frac{d}{dx}\left(f^{(k)}(x)\right) = \frac{d}{dx}\left(p_k(x^{-1})f(x)\right)$$

Using the product rule, we have:

$$f^{(k+1)}(x) = p_k'(x^{-1})f(x) + p_k(x^{-1})f'(x)$$

Now, from the Base Case, we know that $f'(x) = p_1(x^{-1})f(x)$. We can substitue this back into the above equation to get:

$$\begin{aligned} f^{(k+1)}(x) &= p_k'(x^{-1})(x^{-1})'f(x) + p_k(x^{-1})p_1(x^{-1})f(x) \\ &= \left(p_k'(x^{-1})(x^{-1})' + p_k(x^{-1})p_1(x^{-1})\right) f(x) \end{aligned}$$

Let $p_{k+1}(Y) = p_k'(Y) + p_k(Y)p_1(Y)$. This is a polynomial in Y as it is a sum and product of polynomials. Thus, we have $f^{(k+1)}(x) = p_{k+1}(x^{-1})f(x)$.

Thus, by induction, we have shown that for $x \neq 0$ and $n \geq 1$, $f^{(n)}(x)$ is of the form $p_n(x^{-1})f(x)$ for some polynomial $p_n(Y)$ in the variable Y.

Now, we need to show that $f^{(n)}(0)$ exists for all $n \geq 0$.

We know $f(0) = 0$.
Let us proceed by induction on n.
Base Case: For $n = 0$, $f(0) = 0$ exists.
Induction Hypothesis: Assume that for some $k \geq 0$, $f^{(k)}(0)$ exists.
Induction Step: We need to show that $f^{(k+1)}(0)$ exists.
We have:

$$f^{(k+1)}(0) = \lim_{h \to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h \to 0} \frac{f^{(k)}(h)}{h}$$

From the claim, we know that for $h \neq 0$, $f^{(k)}(h) = p_k(h^{-1})f(h)$. Thus,

$$f^{(k+1)}(0) = \lim_{h \to 0} \frac{p_k(h^{-1})f(h)}{h} = \lim_{h \to 0} \frac{p_k(h^{-1})\exp\left(-\frac{1}{h^2}\right)}{h}$$

Here we see that $\exp\left(-\frac{1}{h^2}\right) = \frac{1}{\exp\left(\frac{1}{h^2}\right)}$. So as $h \to 0$, $\frac{1}{h^2} \to \infty$, and therefore, the exponential function grows.

We know, that the exponential function grows much faster than the polynomial function. So, the denominator grows faster than the numerator. Therefore, $f^{(k+1)}(0) \to 0$. □

Proof:
Now, we show that $T_{f,0}$ is identically the 0 function on all of $\mathbb{R}$ and so it cannot be the function f.

$$T_{f,0}(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

From the above proof, we know $f^{(n)}(0) = 0, \forall n$.

$$T_{f,0}(x) = \sum_{n=0}^{\infty} \frac{0}{n!} \cdot x^n = 0$$

□

Problem 5

If the following power series converge, find the radius of convergence (centered at 0):

(1) $\sum_{n=0}^{\infty} n! z^n$ (you are not allowed to use Stirling's approximation),

(2) $\sum_{n=0}^{\infty} c_n z^n$ where $c_n = \begin{cases} 2^n & \text{if } n \text{ is odd,} \\ 3^n & \text{if } n \text{ is even.} \end{cases}$

Solution 5:

Part (1):

Proof: Let $a_n = n!$. We will use the ratio test to find the radius of convergence.
We have:

$$\begin{aligned} \beta &= \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \right| \\ &= \lim_{n \to \infty} (n+1) \\ &= \infty \end{aligned}$$

The radius of convergence R is given by $R = \frac{1}{\beta} \implies R = 0$, for $|z| < 0$.
□

Part (2):

Proof: Let $a_n = c_n$. We will use the root test to find the radius of convergence. We have:

$$\begin{aligned}\beta &= \limsup {a_n}^{1/n} \\ &= \limsup {c_n}^{1/n} \\ &= \limsup (3^n)^{1/n} \\ &= 3\end{aligned}$$

The radius of convergence R is given by $R = \frac{1}{\beta} \implies R = \frac{1}{3}$. □

Problem 6

Consider the polynomial

$$P(z) = \prod_{1 \le k \le 2025} (z - ik).$$

(1) Show that

$$\frac{P'(z)}{P(z)} = \sum_{1 \le k \le 2025} \frac{1}{z - ik} \quad \text{whenever } P(z) \neq 0.$$

(2) Show that roots of P' lie in the upper-half plane.

Solution 6:

Part (1):

Proof:
Claim: $P'(z) = \sum_{k=1}^{2025} (\prod_{j=1, j\neq k}^{2025} (z - ij))$

We will prove this by induction on n.

Base Case: For $n = 1$, we have $P(z) = (z - i)$. Thus, $P'(z) = 1$. The claim holds.
Induction Hypothesis: Assume that for some $k \geq 1$, the claim holds for $n = k$.
Induction Step: We need to show that the claim holds for $n = k + 1$.
We have: $\substack{1 \le j \le 2025 \\ j \neq k}$

$$\begin{aligned}P(z) &= \prod_{j=1}^{k+1} (z - ij) \\ &= (z - i(k+1)) \prod_{j=1}^{k} (z - ij)\end{aligned}$$

Using the product rule, we have:

$$P'(z) = \prod_{j=1}^{k} (z - ij) + (z - i(k+1)) \cdot \frac{d}{dz} \left(\prod_{j=1}^{k} (z - ij) \right)$$

From the induction hypothesis, we know that $\frac{d}{dz} \left(\prod_{j=1}^{k} (z - ij) \right) = \sum_{m=1}^{k} (\prod_{l=1, l\neq m}^{k} (z - il))$. We can substitute this back into the above equation to get:

$$\begin{aligned}P'(z) &= \prod_{j=1}^{k} (z - ij) + (z - i(k+1)) \cdot \sum_{m=1}^{k} (\prod_{l=1, l\neq m}^{k} (z - il)) \\ &= \prod_{j=1}^{k} (z - ij) + \sum_{m=1}^{k} ((z - i(k+1)) \cdot (\prod_{l=1, l\neq m}^{k} (z - il))) \\ &= \sum_{m=1}^{k+1} (\prod_{l=1, l\neq m}^{k+1} (z - il))\end{aligned}$$

Thus, by induction, we have proved the claim.

Now, we can write:

$$\frac{P'(z)}{P(z)} = \frac{\sum_{k=1}^{2025}(\prod_{j=1,j\neq k}^{2025}(z-ij))}{\prod_{j=1}^{2025}(z-ij)}$$
$$= \sum_{k=1}^{2025} \frac{1}{z-ik}$$

whenever $P(z) \neq 0$.
□

Part (2):

Proof: We know, $P(z) = \prod_{k=1}^{2025}(z-ik)$. The roots of $P(z)$ are of the form $z = ik, k \in [1, 2025]$. So, the roots are: $i, 2i, 3i, \ldots, 2025i$. This shows us that all the roots of P are imaginary, and lie in the upper-half plane.

Observe that:

$$P'(z) = 0 \iff \frac{P'(z)}{P(z)} = 0$$

whenever $P(z) \neq 0$.
Any root of $P'(z)$ satisfies root of $\frac{P'(z)}{P(z)} = \sum_{k=1}^{2025} \frac{1}{z-ik}$.
Let $z = x + iy$ where $x, y \in \mathbb{R}$.

$$\frac{1}{z-ik} = \frac{1}{x+iy-ik}$$
$$= \frac{x-i(y-k)}{x^2+(y-k)^2}$$

Now,

$$\sum_{k=1}^{2025} \frac{x-i(y-k)}{x^2+(y-k)^2} = \sum_{k=1}^{2025} \frac{x}{x^2+(y-k)^2} - i\sum_{k=1}^{2025} \frac{(y-k)}{x^2+(y-k)^2}$$

For $\frac{P'(z)}{P(z)} = 0$, both real and imaginary parts have to be equal to 0.

$$0 = \sum_{k=1}^{2025} \frac{(y-k)}{x^2+(y-k)^2}$$

If $y \leq 0$, then $(k-y) > 0$,

$$\sum_{k=1}^{2025} \frac{(y-k)}{x^2+(y-k)^2} > 0$$

But this is not true, as the imaginary part is equal to 0. Thus, $y > 0$.
□

Problem 7

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that

$$f'(z) = f(z), \quad \forall z \in \mathbb{C}.$$

Show that

$$f(z) = f(0)e^z.$$

Hint: Use the result that derivative zero implies function is constant.

Solution 7:

Proof: Let $g(z) = \frac{f(z)}{e^z}$.

$$g'(z) = \frac{f'(z)e^z - f(z)e^z}{e^{2z}}$$

Put $f'(z) = f(z)$ as given in the problem:

$$g'(z) = \frac{f(z)e^z - f(z)e^z}{e^{2z}} = 0$$

We know, that if the derivative of a function is zero, then the function is a constant. Thus, $g(z) = C$, for some $C \in \mathbb{C}$.

$$\begin{aligned} g(z) = C &= \frac{f(z)}{e^z} \\ &\implies \frac{f(z)}{e^z} = C \\ &\implies f(z) = Ce^z \end{aligned}$$

Evaluating f at $z = 0$:

$$f(0) = Ce^0 = C$$

This shows $f(z) = f(0)e^z$. $\square$