**Name:** Anwesha Ghosh
**Class:** MAT-1001-2 (Linear Algebra, Spring 2025)
**Professor:** Dr. Suraj Krishna

## **Assignment – 1**

**Question 1:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as $f(x, y) = \min\{x, y\}$ for all $(x, y) \in \mathbb{R}^2$. Check if this function is linear by the definition then give a proof, or if it is not linear provide an example.

**Solution:**
This map is not linear.
Here is a counterexample.
Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, where

$$(x_1, y_1) = (-1, 3), \quad (x_2, y_2) = (4, -3)$$

Now, according to the definition of linearity, one of the conditions that must hold is:

$$f((x_1, y_1) + (x_2, y_2)) = f(x_1, y_1) + f(x_2, y_2)$$

Substitute the values:

$$\begin{aligned} \text{LHS:} \quad & f((-1, 3) + (4, -3)) = f((3, 0)) = \min\{3, 0\} = 0 \\ \text{RHS:} \quad & f(-1, 3) + f(4, -3) = \min\{-1, 3\} + \min\{4, -3\} \\ & \qquad\qquad\qquad\quad = -1 + (-3) = -4 \end{aligned}$$

$$\therefore \text{LHS} \neq \text{RHS} \quad \Rightarrow \quad f \text{ is not a linear map.}$$

**Question 2:** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined as $f(x, y) = \sqrt{x + y}$ for all $(x, y) \in \mathbb{R}^2$. Check if this function is linear by the definition then give a proof, or if it is not linear provide an example.

**Solution:**
The map $f$ is not linear.
Here is a counterexample:
Let $(x, y) \in \mathbb{R}^2$, let $\alpha = 2$.

According to the definition of linearity, the following condition must be satisfied:

$$\forall (x, y) \in \mathbb{R}^2, \quad \forall \alpha \in \mathbb{R} \setminus \{0\}, \quad f(\alpha(x, y)) = \alpha f(x, y)$$

For $\alpha = 2$:

$$\begin{aligned} &\text{LHS:} \quad f(2(x,y)) = f(2x,2y) = \sqrt{2x+2y} = \sqrt{2(x+y)} = \sqrt{2}\sqrt{x+y} \\ &\text{RHS:} \quad 2f(x,y) = 2\sqrt{x+y} \end{aligned}$$

$$\Rightarrow \text{LHS} \neq \text{RHS} \quad \therefore f \text{ is not linear.}$$

**Question 3:** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function defined as $f(x, y) = (x + y, -x)$ for all $(x, y) \in \mathbb{R}^2$. Check if this function is linear by the definition then give a proof, or if it is not linear provide an example.

**Solution:**
The map $f$ is linear.
**Proof:**
According to the definition, a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is linear if for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$:

1. $\varphi((x_1, y_1) + (x_2, y_2)) = \varphi(x_1, y_1) + \varphi(x_2, y_2)$
2. $\varphi(\alpha(x, y)) = \alpha\varphi(x, y)$

**Additivity:**

$$\begin{aligned} f((x_1, y_1) + (x_2, y_2)) &= f(x_1 + x_2,\ y_1 + y_2) \\ &= ((x_1 + x_2) + (y_1 + y_2),\ -(x_1 + x_2)) \\ &= (x_1 + y_1,\ -x_1) + (x_2 + y_2,\ -x_2) \\ &= f(x_1, y_1) + f(x_2, y_2) \end{aligned}$$

**Homogeneity:**

$$\begin{aligned} f(\alpha(x, y)) &= f(\alpha x,\ \alpha y) \\ &= (\alpha x + \alpha y,\ -\alpha x) \\ &= \alpha(x + y,\ -x) \\ &= \alpha f(x, y) \end{aligned}$$

Both properties are satisfied, so $f$ is a linear map. □

**Question 4:** Consider the following system of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

In matrix form, this can be written as:

$$\mathbf{A}\mathbf{x} = \mathbf{b},$$

where:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Two systems of linear equations are considered **equivalent** if they have the same solution set. Prove the following for the above system:

1. For fixed $i$ and $j$, swapping the $i$-th row with the $j$-th gives a new system which is equivalent to the above system.

**Proof:**

Let the solution set of the given system be $S$. Let the new system be $\mathbf{A}'\mathbf{x} = \mathbf{b}'$, and its solution set be $S'$.

We need to show that $S = S'$.

**(I) To show:** $S \subseteq S'$

Fix $i, j \leq m$ (number of rows).

Swapping the $i$-th row with the $j$-th row in $\mathbf{A}\mathbf{x} = \mathbf{b}$, we get $\mathbf{A}'\mathbf{x} = \mathbf{b}'$.

Now, choose some $\vec{s} \in S$. Then, for all $k \in \{1, \ldots, m\}$,

$$a_{k1}s_1 + a_{k2}s_2 + \cdots + a_{kn}s_n = b_k$$

To show $\vec{s} \in S'$:

- For $k \neq i, j$: $\vec{s}$ satisfies the $k$-th equation of $\mathbf{A}'\mathbf{x} = \mathbf{b}'$.
- For $k = i$: the $i$-th equation in $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ is the $j$-th equation in $\mathbf{A}\mathbf{x} = \mathbf{b}$, which $\vec{s}$ satisfies.
- For $k = j$: the $j$-th equation in $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ is the $i$-th equation in $\mathbf{A}\mathbf{x} = \mathbf{b}$, which $\vec{s}$ satisfies.

Therefore, $\vec{s} \in S'$ for all $\vec{s} \in S$, so $S \subseteq S'$.

**(II) To show:** $S' \subseteq S$

The same argument holds in reverse.

Choose some $\vec{s}' \in S'$. Then, for all $k \in \{1, \ldots, m\}$,

$$a'_{k1}s'_1 + a'_{k2}s'_2 + a'_{kn}s'_n = b'_k$$

To show $\vec{s}' \in S$:

- For $k \neq i, j$: $\vec{s}'$ satisfies the $k$-th equation in $\mathbf{A}\mathbf{x} = \mathbf{b}$.
- For $k = i$: $\vec{s}'$ satisfies the $j$-th equation in $\mathbf{A}\mathbf{x} = \mathbf{b}$.
- For $k = j$: $\vec{s}'$ satisfies the $i$-th equation in $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Therefore, $\vec{s}' \in S$ for all $\vec{s}' \in S'$, so $S' \subseteq S$.

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