Local Dynamics Around Attracting Fixed Points of Meromorphic Maps

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LOCAL DYNAMICS AROUND ATTRACTING FIXED POINTS OF MEROMORPHIC MAPS: A SURVEY

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Abstract. This exposition surveys the local behaviour of meromorphic maps of the Riemann sphere in the neighbourhood of attracting fixed points. After recalling fundamental complex-analytic tools (holomorphicity, normal families, Montel's theorem, Riemann–Hurwitz), it introduces the Fatou and Julia sets, along with stating some properties and their proofs. The article focuses on attracting periodic orbits and explores important results of the same from Koenigs Linearisation and its global extension to studying the behaviour of critical points. Throughout, both rigorous proofs and the geometric intuition in complex dynamics are emphasized.

1. Introduction

Holomorphic dynamics is the study of the reiterates of rational maps on the Riemann sphere $\hat{\mathbb{C}}$. These form the central objects of interest. One begins by classifying points based on their long-term behaviour: some points eventually repeat (preperiodic), some remain fixed, and others wander chaotically.

A fixed point z_0 of a map f satisfies $f(z_0) = z_0$, and periodic points satisfy $f^{\circ n}(z_0) = z_0$ for some $n \geq 1$. The nature of such points is governed by the *multiplier*, defined as the derivative of the appropriate iterate of f at the point: for a fixed point, this is $\lambda = f'(z_0)$, and for a point of *exact* (i.e. it is the smallest period of that point) period n, it is $\lambda = (f^{\circ n})'(z_0)$. This determines whether the point is attracting ($|\lambda| < 1$), repelling ($|\lambda| > 1$), or neutral ($|\lambda| = 1$). We focus on studying local dynamics of geometrically attracting fixed points, that is the fixed points for which the multiplier satisfies $0 < |\lambda| < 1$.

The article is broken down into three sections. The first section lists the main theorems and ideas from complex analysis that are used in the following sections. For this section, we referred to the books of Ahlfors ([3]) and Nevanlinna-Paatero ([2]), and the reader may find the proofs there. The second section sets up the complex dynamic framework. It lists the main structures, ideas and properties of dynamical systems, along with their proofs. We referred to Milnor's book ([1]) for this section. The last section, develops the local fixed point theory for attracting

fixed points. It elicits the main theorems and results, along with their proofs. Milnor's book ([1]) was used for this section too.

2. Complex Analytic Prerequisite

This section contains defintions, theorems and lemmas from Complex Analysis which will be used further in this paper. The proofs can be found in books by Ahlfors ([3]) and Nevalinna-Paatero ([2]).

2.1. **Holomorphic maps.**

Definition 2.1 (Holomorphic, Conformal)**.** If $V \subset \mathbb{C}$ is an open set of complex numbers, a function $f : V \to \mathbb{C}$ is called *holomorphic* (or "complex analytic") if the first derivative

$$z \mapsto f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

is defined and continuous as a function from V to $\mathbb{C}$, or equivalently if f has a power series expansion about any point $z_0 \in V$ which converges to f in some neighbourhood of z_0. (see [3])

Such a function is *conformal* if the derivative $f'(z)$ never vanishes. Thus our conformal maps must always preserve orientation.

2.1.1. *Properties of Holomorphic Maps.*

(1) If two holomorphic maps on $\hat{\mathbb{C}}$ agree on an open set, then they agree everywhere.
(2) *(Liouville's Theoerem)* An entire bounded holomorphic function on $\mathbb{C}$ is constant.
(3) *(Maximum Modulus)* A non-constant holomorphic function on a bounded domain achieves its maximum(in modulus) on the boundary of the domain.
(4) *(Open Mapping theorem)* Any non-constant holomorphic function on a domain is open.

Remark 2.2 (Analytic Continuation)**.** Given a holomorphic function $f : U \to \mathbb{C}$, Given V such that $U \subset V$, $\tilde{f} : V \to \mathbb{C}$ is said to be the analytic continuation of f to V if $\tilde{f}|_U = f$.

Definition 2.3 (Meromorphic functions)**.** A function that is holomorphic at all but finitely many points on a domain D is called meromorphic.

Lemma 2.4 (Schwarz Lemma)**.** *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map with $f(0) = 0$, then the derivative at the origin satisfies $|f'(0)| \leq 1$. If equality holds, $|f'(0)| = 1$,*

then f is a rotation about the origin. That is, $f(z) = cz$ for some constant $c = f'(0)$ on the unit circle. On the other hand, if $|f'(0)| < 1$, then $|f(z)| < |z|$ for all $z \neq 0$. [1]

Definition 2.5 (Normal Family)**.** A family of functions $\mathcal{F}$ is called normal in a domain D if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets of D. [4]

Theorem 2.6 (Weierstrass Uniform Convergence Theorem)**.** *If a sequence of holomorphic functions $f_n : U \to \mathbb{C}$ converges uniformly to the limit function f on compact subsets of U, then f is itself holomorphic. Furthermore, the sequence of derivatives f_n' converges, uniformly on any compact subset of U, to the derivative f'.*[1]

By an inductive argument the same conclusion holds for higher derivatives: for each $k \geq 2$, the sequence $f_n^{(k)}$ converges uniformly on compact subsets of U to $f^{(k)}$. [1]

Theorem 2.7 (Arzelà–Ascoli Theorem)**.** *A family $\mathcal{F}$ of holomorphic functions on an open set $U \subset \mathbb{C}$ is normal if and only if $\mathcal{F}$ is equicontinuous and locally uniformly bounded on U.* [4]

Theorem 2.8 (Montel's Theorem)**.** *Let $S \subseteq \hat{\mathbb{C}}$ and let $\mathcal{F}$ be a collection of holomorphic maps $f : S \to \hat{\mathbb{C}}$ which omit three different values. That is, assume that there are distinct points $a, b, c \in \hat{\mathbb{C}}$ so that*

$$f(S) \subset \hat{\mathbb{C}} \setminus \{a, b, c\}$$

for every $f \in \mathcal{F}$. Then $\mathcal{F}$ is a normal family on S ; that is, the closure $\overline{\mathcal{F}} \subset \mathrm{Hol}(S, \hat{\mathbb{C}})$ is a compact set. [1]

Theorem 2.9 (Riemann-Hurwitz)**.** *Given a meromorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, then it has exactly $2d - 2$ critical points counted with multiplicity.* [4]

Definition 2.10 (Conformal Automorphism)**.** A *conformal automorphism* of the Riemann sphere $\hat{\mathbb{C}}$ is a homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is holomorphic on $\hat{\mathbb{C}}$ and whose inverse f^{-1} is also holomorphic on $\hat{\mathbb{C}}$. In other words, it is a bijective holomorphic self-map of $\hat{\mathbb{C}}$ with holomorphic inverse.

2.2. **Möbius Transformations.**

Definition 2.11 (Möbius transformation)**.** A function $S(z)$ of the form

$$S(z) = \frac{az + b}{cz + d}, \quad \text{with } ad - bc \neq 0,$$

is called a *Möbius transformation* (or *linear fractional transformation*).

The complex numbers $a, b, c, d \in \mathbb{C}$ are called the coefficients of the Möbius transformation. The value $ad - bc$ is called its *determinant*, and must be nonzero.

2.2.1. *Properties of Möbius transformation.*

(1) Any and all conformal automorphisms of $\hat{\mathbb{C}}$ are Möbius transformations. $\mathrm{Aut}(\hat{\mathbb{C}}) = \mathrm{Möb}(\hat{\mathbb{C}})$.
(2) Möbius transformations on $\hat{\mathbb{C}}$ can send a triplet of points to any other triplet of points.
(3) A Möbius transformation has almost 2 fixed points.
(4) A Möbius transformation is uniquely determined by fixing its value at three distinct points.

3. Complex Dynamical Prerequisite

3.1. **Fatou and Julia Sets.** Consider $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a nonconstant holomorphic map. The sequence $\{f^{\circ n}\}$ is defined as $f^{\circ 1} = f$, and $f^{\circ n} = f \circ f^{\circ(n-1)}$ for $n \geq 2$, or the n^{th} iterate of f.

Definition 3.1 (Fatou Set)**.** The Fatou set $\mathcal{F}_f$ is the set of all points, z such that $\{f^{\circ n}\}$ is normal at z.

Definition 3.2 (Julia Set)**.** The Julia set $\mathcal{J}_f$ is the complement of the Fatou set, i.e., $\mathcal{J}_f = \hat{\mathbb{C}} \setminus \mathcal{F}_f$.

Remark 3.3. The Fatou set is open and the Julia set is closed.

3.2. **Properties of Fatou and Julia Sets.**

Lemma 3.4 (Invariance Lemma)**.** *The Julia set $\mathcal{J} = \mathcal{J}(f)$ of a holomorphic map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is fully invariant under f. That is, z belongs to $\mathcal{J}$ if and only if the image $f(z)$ belongs to $\mathcal{J}$. Equivalently, the Fatou set is fully invariant.* [1]

$$f^{-1}(\mathcal{J}) = \mathcal{J} = f(\mathcal{J})$$
$$f^{-1}(\mathcal{F}) = \mathcal{F} = f(\mathcal{F})$$

Proof. We have to show that $z \in \mathcal{J} \iff f(z) \in \mathcal{J}$.

$\Rightarrow$ If $z \in \mathcal{J}$, then $\{f^{\circ n}\}$ is not normal at z. By the definition of normality (2.5), there exists a subsequence $\{f^{\circ n_k}(z)\}$ that does not converge locally uniformly to any point in $\hat{\mathbb{C}}$. Since f is continuous, the image of this subsequence, $\{f(f^{\circ n_k}(z))\} = \{f^{\circ(n_k+1)}(z)\}$, also does not converge locally uniformly to any point in $\hat{\mathbb{C}}$. Thus, $f(z) \in \mathcal{J}$.
$\Leftarrow$ If $f(z) \in \mathcal{J}$, then by the same reasoning, we can show that $\{f^{\circ n}(f(z))\}$ does not converge to any point in $\hat{\mathbb{C}}$. Hence, $z \in \mathcal{J}$.

□

Lemma 3.5 (Iteration Lemma)**.** *For any $k > 0$, the Julia set $\mathcal{J}(f^{\circ k})$ of the k-fold iterate coincides with the Julia set $\mathcal{J}(f)$. Equivalently, the Fatou set $\mathcal{F}(f^{\circ k})$ of the k-fold iterate coincides with the Fatou set $\mathcal{F}(f)$.*[1]

Proof. We will show that $\mathcal{F}(f^{\circ k}) = \mathcal{F}(f)$.
First, we will show that $\mathcal{F}(f) \subset \mathcal{F}(f^{\circ k})$. Let $z \in \mathcal{F}(f)$. By definition, there exists an open neighbourhood $U \ni z$ such that $\{f^{\circ n}\}$ is a normal family. It follows that any subfamily say, $\{f^{\circ kn}\}$ is also normal on U. This is exactly the iterates of the map $f^{\circ k}$, as:

$$(f^{\circ k})^n = f^{\circ kn}$$

This implies that $\{f^{\circ k}\}$ is a normal family on U, which means $z \in \mathcal{F}(f^{\circ k})$.

Now, we show that $\mathcal{F}(f^{\circ k}) \subset \mathcal{F}(f)$. Let a point $z \in \mathcal{F}(f^{\circ k})$. By definition, the family of itereates of $f^{\circ k}$, $\{f^{\circ kn}\}_{n\in\mathbb{N}}$ is normal at z. This means that there exists an compact neighbourhood $U \ni z$ such that every subsequence of $\{f^{\circ kn}\}$ has a further subsequence that converges uniformly on U.
We will show that the family $\{f^{\circ n}\}_{n\in\mathbb{N}}$ is normal on V, which we construct later. Let $f^{\circ n_l}$ be a subsequence of $\{f^{\circ n}\}$. Then, we can write $n_l = km_l + r_l$, where $0 \leq r_l < k$. Also, since the residue classes modulo k are finite, we can extract a further subsequence $\{f^{\circ(km_l+r)}\}_{l\in\mathbb{N}}$ such that $r_l = r$ for all l. Now, Let $V = f^{-r}(U)$. Since $f^{\circ r}$ is continuous, the image of U under f^{-r} is an open neighbourhood of z. For all $z' \in V$, we have $f^{\circ r}(z') \in U$. So the family $\{f^{\circ(km_l+r)}\}_{l\in\mathbb{N}}$ on V converges uniformly, as it is the same as the family $\{f^{\circ km_l}\}_{l\in\mathbb{N}}$ on U. Thus, we have shown that the family $\{f^{\circ n}\}_{n\in\mathbb{N}}$ is normal on V, which means $z \in \mathcal{F}(f)$.

Therefore, $\mathcal{F}(f^{\circ k}) = \mathcal{F}(f)$. □

Definition 3.6 (Periodic Point)**.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. A point $z \in \hat{\mathbb{C}}$ is called a *periodic point* of f if there exists an integer $n \geq 1$ such that

$$f^{\circ n}(z) = z.$$

The smallest such n is called the *period* of z.

Definition 3.7 (Multiplier)**.** Consider a map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with a period m. The first derivative of the m-fold iterate $f^{\circ m}$ at a point of the orbit is a well-defined complex number called the *multiplier* of the orbit. The multiplier is commonly denoted by λ, so we have $\lambda = (f^{\circ m})'(z)$.

Remark 3.8. Periodic orbits are classified into different kinds based on the value of the multiplier, as follows:

- $|\lambda| < 1 \implies$ attracting

- $\diamond$ $|\lambda| = 0 \implies$ super–attracting
- $\diamond$ $0 < |\lambda| < 1 \implies$ geometrically attracting

- $|\lambda| > 1 \implies$ repelling
- $|\lambda| = 1 \implies$ neutral
  - $\diamond$ rational or parabolic, if λ is a root of unity
  - $\diamond$ irrational, otherwise.

Remark 3.9. The study of periodic orbits can be reduced to a study of fixed points, as an m–periodic point of f is a fixed point of $f^{\circ m}$, and Iteration Lemma 3.5 tells us that the Julia and Fatou sets of the two are same.

Definition 3.10 (Basin of Attraction)**.** If $\mathcal{O}$ is an attracting periodic orbit of period m, then we define the *basin of attraction* to be the open set $\mathcal{A} \subset \hat{\mathbb{C}}$ consisting of all points $z \in \hat{\mathbb{C}}$ for which successive iterates $f^{\circ m}(z), f^{\circ 2m}(z), \dots$ converge towards some point of $\mathcal{O}$.

Consider an attracting fixed point, and N_0 a neighbourhood of it. Since we know that the forward orbit of every point in $\mathcal{A}$ is eventually contained in N_0, $\mathcal{A} = \bigcup_{k=0}^{\infty} f^{\circ(-k)}(N_0)$.

Lemma 3.11. *Every attracting periodic orbit is contained in the Fatou set of f. In fact, the entire basin of attraction $\mathcal{A}$ for an attracting periodic orbit is contained in the Fatou set. However, every repelling periodic orbit is contained in the Julia set.*

Proof. First, we will show that every repelling periodic orbit is contained in the Julia set.

Consider the non-constant holomorphic map, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Take $F = f^{\circ m}$. Let p be the fixed point, so $F(p) = p$. Let $|\lambda| > 1$, and we know $F'(p) = (f^{\circ m})'(p) = \lambda$.

Assume for the sake of contradiction, $p \in \mathcal{F}(F)$. By the definition of Fatou set, $\exists$ an open neighbourhood $U \ni p$ such that $\{F^{\circ n}\}$ is normal on U.

Choose a small closed disk $\overline{\mathbb{D}} \subset U \ni p$. Since $\overline{\mathbb{D}}$ is compact and contained in U, by Theorem 2.7, we know that normality of $\{F^{\circ n}\}$ on U implies equicontinuity on $\overline{\mathbb{D}}$.

By Cauchy's Estimate Theorem, , $\forall z \in \overline{\mathbb{D}}$, we can say there exists an M which bounds $|(F^{\circ n})'|$, that is,

$$|(F^{\circ n})'| \leq M, \forall n \geq 0. \tag{3.1}$$

Since $F(p) = p$, we apply chain rule as follows:

$$\begin{aligned}(F^{\circ n})'(p) &= F'(F^{\circ(n-1)}(p)) \cdots F'(p) \\ &= F'(F^{\circ(n-1)}(p) \cdots \lambda \\ &= \lambda^n\end{aligned}$$

$|\lambda| > 1 \implies \lambda^n \to \infty$ as $n \to \infty$. Now, this implies, as $n \to \infty$:

$$|(F^{\circ n})'| \to \infty \tag{3.2}$$

Equations (3.1) and (3.2) pose a contradiction. Therefore, $p \notin \mathcal{F}(F)$. By Lemma 3.5, we know that $\mathcal{F}(F) = \mathcal{F}(f)$. Hence, $p \notin \mathcal{F}(f)$. This implies $p \in \mathcal{J}(f)$. Thus, we conclude that every repelling periodic orbit is contained in the Julia set.

Now, we will show that every attracting period orbit is contained in the Fatou set. Take $|\lambda| < 1$, and p as an attracting fixed point of the map $F = f^{\circ m}$. The claim is that $p \in \mathcal{F}(F)$. From the same construction as in the proof of Theorem 5.2, there exists a neighbourhood of p on which the iterates of F converge to the constant p function. So $p \in \mathcal{F}(F) = \mathcal{F}(f)$ by Lemma 3.5.

□

Lemma 3.12. *Every parabolic point belongs to the Julia set.*

Proof. Assume for the sake of contradiction, that every parabolic point belongs to the Fatou set, that is, let z_0 be a parabolic point and $z_0 \in \mathcal{F}_f$. Consider the non-constant holomorphic map, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. WLOG (Remark), take 0 as the parabolic fixed point, that is, $f(0) = 0$.
Since, $z_0 = 0 \in \mathcal{F}_f$, we can say that there exists a neighbourhood $U \ni z_0$ where $\{f^{\circ n}\}_{n\in\mathbb{N}}$ forms a normal family on U around the origin. Note that, since z_0 is a parabolic fixed point, $|\lambda| = \zeta_n$ where λ is the multiplier.
Claim: Any subsequence of $\{f^{\circ n}\}_{n\in\mathbb{N}}$ does not uniformly converge on some compact subset of $\hat{\mathbb{C}}$.

> For sake of contradiction, assume that for every sequence $(f^{\circ n_k})_{k\in\mathbb{N}}$ there is a further subsequence $(f^{\circ n_{k_j}})_{j\in\mathbb{N}}$ uniformly converges on $T_{compact} \subseteq \hat{\mathbb{C}}$, any compact subset. Let us break down what we are doing:

$$f(z) = \lambda z + a_2 z^2 + a_3 z^3 + \dots \tag{3.3}$$

Now, computing $f \circ f = f^{\circ 2}(z)$:

$$\begin{aligned}
&(f \circ f)(z) \\
&= f(f(z)) \\
&= f(\lambda z + a_2 z^2 + a_3 z^3 + \ldots) && \text{(from Equation (3.3))} \\
&= \lambda(\lambda z + a_2 z^2 + a_3 z^3 + \ldots) + a_2(\lambda z + a_2 z^2 + a_3 z^3 + \ldots)^2 + \ldots \\
&= \lambda^2 z + \text{(some other terms)}
\end{aligned}$$

Inductively, we see that:

$$f^{\circ n}(z) = \lambda^n z + a_q z^q + a_{q+1} z^{q+1} + \ldots \tag{3.4}$$

Now, we know that $|\lambda|$ is a n^{th} root of unity. Thus, $\lambda^n = 1$ and we can simplify Equation (3.4) as follows:

$$f^{\circ n}(z) = z + a_q z^q + a_{q+1} z^{q+1} + \ldots \tag{3.5}$$

From Remark 3.9, we can now look at the map $f^{\circ n} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. As we did above, we compute $(f^{\circ n})^{\circ 2}$:

$$\begin{aligned}
&(f^{\circ n} \circ f^{\circ n})(z) \\
&= f^{\circ n}(f^{\circ n}(z)) \\
&= f^{\circ n}(z + a_q z^q + a_{q+1} z^{q+1} + \ldots) \\
&= (z + a_q z^q + a_{q+1} z^{q+1} + \ldots) + a_q(z + a_q z^q + a_{q+1} z^{q+1} + \ldots)^q + \ldots \\
&= z + 2a_q z^q + \text{(some other terms)}
\end{aligned}$$

Let us look at the p^{th} iterates of this map. Using an inductive argument, we see that:

$$(f^{\circ n})^{\circ p} = z + p a_q z^q + \ldots$$

Now, by Theorem 2.6, the q^{th} derivative of every subsequence of $(f^{\circ n})^{\circ p}$ should converge locally uniformly. However, taking the q^{th} derivative of $(f^{\circ n})^{\circ p}$ around 0, we get:

$$(f^{\circ np})^{(q)}(0) = q! p a_q$$

Since, $q! p a_q \to \infty$, no subsequence of $\{f^{\circ np_j}\}$ can converge locally uniformly as $k_j \to \infty$. Hence, this is a contradiction. Therefore, we have proved our claim.

Thus, no subsequence of $\{f^{\circ n}\}_{n \in \mathbb{N}}$ converges locally uniformly. $\{f^{\circ n}\}_{n \in \mathbb{N}}$ does not form a normal family. This is a contradiction. Therefore, $p \notin \mathcal{F}_f \iff p \in \mathcal{J}_f$.

□

Lemma 3.13. *If f is rational of degree 2 or more, then the Julia set is nonvacuous.*

Proof. Assume for contradiction $\mathcal{J}_f$ is vacuous. This implies that all iterates of f belong to the Fatou set. We know that there must be a sequence of iterates, say $f^{\circ n_j}$, which converge uniformly over entirety of $\hat{\mathbb{C}}$ to a holomoprhic limit (Theorem 2.6), $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Since f is a rational map of degree, $d \geq 2$, there exists a large j such that the degree of $f^{\circ n_j}$ is equal to the degree of g. Now, we know the degree of $f^{\circ n}$ is d^n, and $d^n \to \infty$ as $n \to \infty$. g being holomorphic is of finite degree, which is a contradiction if its degree should be equal to degree of $f^{\circ n_j}$ for some j, as that is infinity. Therefore, $\mathcal{J}_f$ is nonvacuous.

□

Definition 3.14 (Forward Orbit)**.** Consider the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The *forward orbit* of a point $z \in \hat{\mathbb{C}}$ is the set O^+ that consists of points $z, f(z), f^2(z), \ldots$.

Definition 3.15 (Backward Orbit)**.** Consider the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The *backward orbit* of a point $z \in \hat{\mathbb{C}}$ is the set O^- that consists of points $z, f^{-1}(z), f^{-2}(z), \ldots$.

Definition 3.16 (Grand Orbit)**.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map. The *grand orbit* of a point $z \in \hat{\mathbb{C}}$ under f is the set

$$\mathrm{GO}(z, f) := \left\{ z' \in \hat{\mathbb{C}} \,\middle|\, f^{\circ m}(z) = f^{\circ n}(z') \text{ for some } m, n \geq 0 \right\}.$$

That is, $z' \in \mathrm{GO}(z, f)$ if the forward orbit of z' eventually intersects the forward orbit of z.

Two points $z, z' \in \hat{\mathbb{C}}$ belong to the same grand orbit if and only if there exist integers $m, n \geq 0$ such that

$$f^{\circ m}(z) = f^{\circ n}(z').$$

Definition 3.17 (Exceptional Set)**.** A point $z \in \hat{\mathbb{C}}$ is called *exceptional* under f if its grand orbit is finite:

$$|\mathrm{GO}(z, f)| < \infty.$$

In other words, z is exceptional if only finitely many points in $\widehat{\mathbb{C}}$ eventually map to the orbit of z.

Lemma 3.18. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is rational of degree $d \geq 2$, then the set $\mathcal{E}(f)$ of grand orbit finite points can have at most two elements. These grand orbit finite points, if they exist, must always be superattracting periodic points of f and hence must belong to the Fatou set.*

Proof. A point $z \in \hat{\mathbb{C}}$ belongs to $\mathcal{E}(f)$ if its grand orbit is finite; that is, there are only finitely many points $w \in \hat{\mathbb{C}}$ such that some iterate of f maps w to an iterate of z. This means both the forward and backward orbits of z are finite.

Observe that if $z \in \mathcal{E}(f)$, then:

- Any forward image $f^n(z)$ also has a finite grand orbit, since forward images of a finite set remain finite.
- Any preimage $w \in f^{-1}(z)$ must also be in $\mathcal{E}(f)$. Otherwise, infinitely many distinct points would eventually map to z, contradicting the finiteness of its grand orbit.

So $\mathcal{E}(f)$ is closed under forward and backward iteration, or it is invariant under f.

Suppose now that $\mathcal{E}(f)$ contains more than two points. We claim this leads to a contradiction with the Riemann–Hurwitz formula (Theorem 2.9). Recall the formula for a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$:

$$\sum_{z \in \hat{\mathbb{C}}} (e_z - 1) = 2d - 2,$$

where e_z is the local degree of f at z. This puts an upper bound on the total amount of ramification a rational map can have.

Now, for a point to have a finite backward orbit (i.e., be in $\mathcal{E}(f)$), the preimages of that point must eventually repeat or collapse, meaning f must be ramified over that point. In particular, if $f^{-1}(z) \subset \{z\}$, then f must have local degree $e_z = d$ at z, contributing $d - 1$ to the ramification sum.

Suppose there are three such exceptional points. Then the total ramification contribution is at least

$$3(d-1) > 2d - 2,$$

which contradicts the Riemann–Hurwitz formula.

But if there are only one or two exceptional points, the total ramification required can be at most $2(d-1) = 2d - 2$, which exactly matches the Riemann–Hurwitz bound.

Therefore, $\mathcal{E}(f)$ has at most two elements.

Finally, every point in $\mathcal{E}(f)$ must be eventually periodic (since its forward orbit is finite), and if it has only finitely many preimages, then those preimages must eventually coincide. This implies a local degree greater than one, so the point is *superattracting*. Hence, it lies in the Fatou set.

□

Theorem 3.19. *Let z_1 be an arbitrary point of the Julia set $\mathcal{J}(f) \subset \hat{\mathbb{C}}$ and let N be an arbitrary neighbourhood of z_1. Then the union U of the forward images*

$f^{\circ n}(N)$ contains the entire Julia set and contains all but at most two points of $\hat{\mathbb{C}}$. More precisely, if N is sufficiently small, then U is the complement $\hat{\mathbb{C}} \setminus \mathcal{E}(f)$ of the set of grand orbit finite points.

Proof. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree ≥ 2. Fix any point $z_1 \in J(f)$ and any open neighbourhood N of z_1. Define

$$U = \bigcup_{n \geq 0} f^{\circ n}(N).$$

Suppose for contradiction, that $\hat{\mathbb{C}} \setminus U$ contains three distinct points a, b, c. Then the family of iterates $\{f^{\circ n}\}$ omits these three values on N, and hence on all of U. According to Montel's Theorem 2.8, such a family is normal. Since $f(U) \subset U$, this implies the entire orbit stays in a normal family, so $U \subset \mathcal{F}(f)$. But $z_1 \in N \subset U$, and $z_1 \in \mathcal{J}(f)$, a contradiction. Hence, $|\hat{\mathbb{C}} \setminus U| \leq 2$.

Let $w \in \hat{\mathbb{C}} \setminus U$. Then $w \notin f^{\circ n}(N)$ for all $n \geq 0$, so no forward iterate of a point in N maps to w. Thus all preimages of w must also lie in $\hat{\mathbb{C}} \setminus U$, and hence inductively all elements of the grand orbit of w lie in $\hat{\mathbb{C}} \setminus U$, which is finite. So $w \in \mathcal{E}(f)$, and $\hat{\mathbb{C}} \setminus U \subset \mathcal{E}(f)$.

Since $\mathcal{E}(f) \subset F(f)$, and $\mathcal{J}(f) \subset \hat{\mathbb{C}} \setminus \mathcal{E}(f)$, we can conclude $\mathcal{J}(f) \subset U$. If $N \cap \mathcal{E}(f) = \emptyset$, then no exceptional point ever enters any $f^{\circ n}(N)$, so $U = \hat{\mathbb{C}} \setminus \mathcal{E}(f)$. □

Corollary 3.20. *If the Julia set contains an interior point, then it must be equal to the entire Riemann sphere.*

Proof. Let $w \in \mathcal{J}_f$. Now, choose a neighbourhood $N \subset \mathcal{J}_f$ of w. Define $U = \bigcup_{n \geq 0} f^{\circ n}(N)$. We know, $U \subset J$ and since $\overline{U} = \hat{\mathbb{C}}$, U is everywhere dense. Since $\mathcal{J}_f$ is a closed set (Remark 3.3), $\mathcal{J}_f = \hat{\mathbb{C}}$. b □

Corollary 3.21. *If $\mathcal{A} \subset \hat{\mathbb{C}}$ is the basin of attraction for some attracting periodic orbit, then the topological boundary $\partial\mathcal{A} = \overline{\mathcal{A}} \setminus \mathcal{A}$ is equal to the entire Julia set. Every connected component of the Fatou set $\hat{\mathbb{C}} \setminus \mathcal{J}$ either coincides with some connected component of this basin $\mathcal{A}$ or else is disjoint from $\mathcal{A}$.*

Proof. First, let's show $\partial\mathcal{A} \subset J(f)$. Let $z \in \partial\mathcal{A}$. Then every neighbourhood $U \ni z$ meets both $\mathcal{A}$ and $\hat{\mathbb{C}} \setminus \mathcal{A}$. If $z \in \mathcal{F}(f)$, the family $\{f^n\}$ is normal on some U, so its limit on $U \cap \mathcal{A}$ would extend to all of U, forcing $U \subset \mathcal{A}$, a contradiction. Hence $z \notin \mathcal{F}(f)$, i.e. $z \in \mathcal{J}(f)$.

Now, let's show $\mathcal{J}(f) \subset \partial\mathcal{A}$. Let $z \in J(f)$. By Theorem 3.19 , for any neighbourhood $N \ni z$,

$$\bigcup_{n\geq 0} f^{\circ n}(N) \;=\; \hat{\mathbb{C}} \setminus \mathcal{E}(f),$$

and since $\mathcal{E}(f) \subset \mathcal{F}(f)$, this union lies in $\mathcal{A}$. Thus some $f^{\circ n}(N)$ meets $\mathcal{A}$, whenever $N \cap \mathcal{A} \neq \emptyset$. Because $z \notin \mathcal{A}$, every N also meets $\hat{\mathbb{C}} \setminus \mathcal{A}$, so $z \in \partial\mathcal{A}$.

Thus, $\partial\mathcal{A} = \mathcal{J}(f)$.

Now, since $\mathcal{F}(f)$ is open and decomposes into its connected components, and $\mathcal{A} \subset \mathcal{F}(f)$, any Fatou component intersecting $\mathcal{A}$ must lie entirely in $\mathcal{A}$, while any disjoint component remains disjoint. □

Corollary 3.22. *If z_0 is any point of the Julia set $\mathcal{J}(f)$, then the set of all iterated preimages*

$$\Big\{ z \in \hat{\mathbb{C}} \,;\; f^{\circ n}(z) = z_0 \text{ for some } n \geq 0 \Big\}$$

is everywhere dense in $\mathcal{J}(f)$.

Proof. Let $P = \Big\{ z \in \hat{\mathbb{C}} \,;\; f^{\circ n}(z) = z_0 \text{ for some } n \geq 0 \Big\}$.
We know by Lemma 3.18, points in the Julia set are never exceptional, that is, $z_0 \notin \mathcal{E}(f)$. Theorem 3.19 implies that for any point $w \in \mathcal{J}(f)$ and any neighbourhood $U \ni w$,

$$\bigcup_{n\geq 0} f^{\circ n}(U) \;=\; \hat{\mathbb{C}} \setminus \mathcal{E}(f) \;\supset\; \{z_0\}.$$

Hence there exists $n \geq 0$ and $z \in U$ with

$$f^{\circ n}(z) = z_0.$$

Thus every neighbourhood U of every $w \in \mathcal{J}(f)$ contains some $z \in P$. Therefore, P is dense in $\mathcal{J}(f)$. □

Corollary 3.23. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Then the Julia set $J(f)$ is perfect: it is closed and has no isolated points.*

Proof. (1) $\mathcal{J}(f)$ is infinite. If $\mathcal{J}(f)$ were finite, then every point of $\mathcal{J}(f)$ would have a finite grand orbit, contradicting Lemma 3.18 (which asserts no Julia point is exceptional). Thus ($J(f)$ contains infinitely many points.

(2) Existence of a limit point. Since $\mathcal{J}(f) \subset \hat{\mathbb{C}}$ is an infinite subset of the compact sphere, it must have at least one accumulation point $z_0 \in \mathcal{J}(f)$.

(3) Density of iterated preimages. By Corollary 3.22, the set of all iterated preimages of z_0,

$$\big\{ z \in \hat{\mathbb{C}} : f^{\circ n}(z) = z_0 \text{ for some } n \geq 0 \big\},$$

is dense in $\mathcal{J}(f)$. In particular, in every neighbourhood of any point of $\mathcal{J}(f)$ there lies some preimage of z_0, hence another distinct point of $\mathcal{J}(f)$.

Let $w \in \mathcal{J}(f)$ be arbitrary and $U \ni w$ any neighbourhood. By density of the preimages of z_0, U contains some $z \neq w$ with $f^{\circ n}(z) = z_0$. Thus w is a limit point of other Julia points, so w is not isolated. Since w was arbitrary, $\mathcal{J}(f)$ has no isolated points. □

Corollary 3.24. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree ≥ 2. Then for a* generic *point $z \in \mathcal{J}(f)$, the forward orbit*

$$\{\, z,\ f(z),\ f^{\circ 2}(z),\ \dots \}$$

is dense in $\mathcal{J}(f)$.

Proof. For each integer $j \geq 1$, choose a finite cover of the Julia set $\mathcal{J}(f)$ by spherical balls $\{N_{j,k}\}_k$ each of diameter $< 1/j$. Define

$$U_{j,k} \;=\; \bigcup_{n \geq 0} f^{-n}\big(N_{j,k}\big),$$

the set of all points whose forward orbit meets $N_{j,k}$. By Corollary 3.22, the iterated preimages of any point in $N_{j,k}$ are dense in $\mathcal{J}(f)$, so $\overline{U_{j,k}} \cap \mathcal{J}(f) = \mathcal{J}(f)$. Hence $U_{j,k} \cap \mathcal{J}(f)$ is a dense open subset of $\mathcal{J}(f)$. Let

$$U_j = \bigcap_k U_{j,k},$$

a finite intersection of dense opens, so U_j is dense and open in $\mathcal{J}(f)$. Finally set

$$U = \bigcap_{j=1}^{\infty} U_j,$$

a countable intersection of dense opens in $\mathcal{J}(f)$, thus itself dense (Baire Category Theorem).

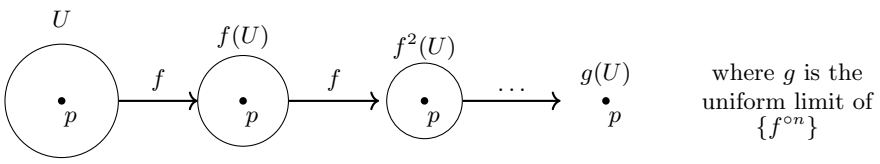
If $z \in U$, then for every j and every covering ball $N_{j,k}$ there is some n with $f^{\circ n}(z) \in N_{j,k}$. Since the $N_{j,k}$ form arbitrarily fine covers of $\mathcal{J}(f)$, the forward orbit of z is dense in $\mathcal{J}(f)$. □

4. Notations

Symbol	Description
$f^{\circ n}$	f composed with itself, n times
$\hat{\mathbb{C}}$	Riemann Sphere
$\mathbb{D}_r$	Disk of radius r around 0
$\mathcal{J}(f)$	Julia set of f
$\mathcal{F}(f)$	Fatou set of f
$\mathcal{E}(f)$	Exceptional set of f
$GO(z, f)$	Grand Orbit of z under f
$\mathcal{A}$	Basin of attraction
$\mathcal{A}_0$	Immediate basin of attraction
∂U	Boundary of U
$\mathcal{N}(z)$	Neighborhood of z
$\mathrm{Hol}(U, V)$	Holomorphic maps from U to V

5. Attracting Fixed points

Definition 5.1. Given a map f and a fixed point p, we call the point *topologically attracting*, if there exists a neighbourhood U of p, such that the family of functions $\{f^{\circ n}\}_{n\in\mathbb{N}}$ converges uniformly to the constant function $g(z) = p$ on U.



One can think of this as a point, which has a neighbourhood which is shrunk down further and further under repeated action by f. After infinite iterations of f, this neighbourhood will be shrunk down to just the point. To characterize this, we use the multiplier.

Lemma 5.2. *A fixed point p for a holomorphic function f is topologically attracting if and only if its multiplier λ satisfies $|\lambda| < 1$*

Proof. ($|\lambda| < 1 \implies$ *topologically attracting*)
First, recall that through Möbius conjugations, we can send p to 0, argue there, and apply the inverse transformation to bring the arguments back to p. We aim to find a neighbourhood that f contracts geometrically. using that, we can show that

the family of iterates of f will converge geometrically to the constant 0 function. So, we show that $\exists r$ such that $\forall z \in \mathbb{D}_r$

$$f(z) \leq c \cdot |z|$$

where $c < 1$. Choose any c such that $|\lambda| < c < 1$.

WLOG, let $f(0) = 0$. Taylor's theorem tells us that for some small enough r_0, $f(z) = \lambda z + O(z^2)$ on $\mathbb{D}_{r_0}$. That is,

$$\begin{aligned} &|f(z) - \lambda z| \leq c_0 \cdot |z^2| && \forall z \in \mathbb{D}_{r_0} \\ &|f(z)| \leq |\lambda z| + c_0 \cdot |z^2| = (|\lambda| + c_0|z|) \cdot |z| && (5.1) \end{aligned}$$

Since this holds on $\mathbb{D}_{r_0}$, it also holds on $\mathbb{D}_r$ for all $r \leq r_0$. By the Archimedean Principle, we know that there exists a small enough r_1 such that $0 < c_0 \cdot r < c - |\lambda| \implies |\lambda| + c_0 \cdot r < c$. Setting $r = \min\{r_0, r_1\}$ and continuing from Equation (5.1) we can say that,

$$|f(z)| \leq (|\lambda| + c_0 \cdot r) \cdot |z| < c \cdot |z| \qquad \forall z \in \mathbb{D}_r$$

Now, the claim is that on this neighbourhood $\mathbb{D}_r$ of 0, the family of functions $\{f^{\circ n}\}$ converges uniformly. So, $\forall z \in \mathbb{D}_r$,

$$|f(z)| < c \cdot |z| < c \cdot r$$

since $c < 1$, $f(z)$ is in $\mathbb{D}_r$. So we can also say $\forall z \in \mathbb{D}_r$

$$|f \circ f(z)| < c \cdot |f(z)| < c^2 \cdot r$$

Continuing this argument,

$$|f^{\circ n}(z)| < c^n |z| < c^n r$$

This converges uniformly to 0 as $n \to \infty$. So, 0 is topologically attracting.

(*topologically attracting* $\implies |\lambda| < 1$)
If 0 is a topologically attracting fixed point, then there exists a neighbourhood U of 0 such that $\{f^{\circ n}\}$ converges uniformly to the constant 0 function on U. So, for all $\epsilon > 0$ there exists an $N > 0$ such that $\forall n > N$,

$$|f^{\circ n}(z)| < \epsilon \qquad \forall z \in U$$

Consider the disk $\mathbb{D}_\epsilon \subset U$. The above statement implies that $\mathbb{D}_\epsilon$ is mapped onto a proper subset of itself by some iterate $f^{\circ n}$. Using Schwarz Lemma (Lemma 2.4) on this iterate and $\mathbb{D}_\epsilon$, we get that the multiplier of $f^{\circ n}$, which is λ^n satisfies $|\lambda^n| < 1$. But this implies that $|\lambda| < 1$.

□

Theorem 5.3 (Koenigs Linearisation Theorem)**.** *Given a geometrically attracting fixed point or a repelling fixed point of f, there exists a local holomorphic change of co-ordinates ϕ, such that $\phi \circ f \circ \phi^{-1}(w) = \lambda w$ for all w in some neighbourhood of 0. Further, ϕ is unique upto multiplication by a constant.*

A restatement of the above, is that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{D}_\epsilon & \xrightarrow{w \mapsto \lambda w} & \mathbb{D}_\epsilon \end{array}$$

This diagram gives the functional equation

$$\lambda \cdot \phi(z) = \phi \circ f(z)$$

Proof. **Existence:** First we show that such a function exists. It is enough to show this for f when $|\lambda| < 1$. This is because for the other case, when $|\lambda| > 1$, we can apply the same argument on f^{-1}, which can be defined locally as a holomorphic function. Note that f^{-1} has the multiplier as $\frac{1}{\lambda}$ and $0 < |\frac{1}{\lambda}| < 1$ Define the following family of functions, $\{\phi_n\}$ as

$$\phi_n(z) = \frac{f^{\circ n}(z)}{\lambda^n}$$

Now it is easy to see from the equation form that the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} \\ \mathbb{D}_\epsilon & \xrightarrow{w \mapsto \lambda w} & \mathbb{D}_\epsilon \end{array}$$

$$\lambda \cdot \circ\, \phi_{n+1}(z) = \lambda \cdot \frac{f^{\circ n+1}(z)}{\lambda^{n+1}} = \frac{f^{\circ n+1}(z)}{\lambda^n} = \frac{f^{\circ n}(f(z))}{\lambda^n} = \phi_n \circ f(z)$$

Now we show that the family of functions $\{\phi_n\}$ converges to a uniform limit. Then, we can take limits as $n \to \infty$ in the above diagram, and obtain our desired commutative diagram, with ϕ being the uniform limit of the family $\{\phi_n\}$.

We know that $|\lambda| < 1$. Choose a constant c such that $c^2 < |\lambda| < c < 1$. From the same argument as in the proof of Lemma 5.2, we know that there exists an open disk $\mathbb{D}_r$ such that for all $z \in \mathbb{D}_r$, $|f(z)| < c|z|$. Consider any $z_0 \in \mathbb{D}_r$. And let z_n denote $f^{\circ n}(z_0)$, the n^{th} element in the forward orbit of z_0. We know that the entire forward orbit of z_0 lies inside the disk, and that

$$|z_n| < r \cdot r^n$$

By Taylor's theorem, we know that in a small neighbourhood around 0 (WLOG, say $\mathbb{D}_r$), $f(z) = \lambda z + O(z^2)$. That is, for some $\alpha > 0$

$$|f(z) - \lambda z| < \alpha |z^2| \qquad , \forall z \in \mathbb{D}_r$$

Now using this relation on z_n, we get,

$$|f(z_n) - \lambda z_n| < \alpha |z_n^2| < \alpha r^2 c^{2n}.$$

Dividing both sides by λ^{n+1}

$$\left| \frac{f(z_n)}{\lambda^{n+1}} + \lambda \cdot \frac{z_n}{\lambda^{n+1}} \right| < \alpha \frac{r^2 c^{2n}}{|\lambda|^{n+1}}$$

$$\left| \frac{f(f^{\circ n}(z))}{\lambda^{n+1}} + \frac{f^{\circ n}(z)}{\lambda^n} \right| < \alpha \frac{r^2}{|\lambda|} \left(\frac{c^2}{|\lambda|} \right)^n$$

$$|\phi_{n+1}(z) + \phi_n(z)| < \frac{\alpha r^2}{|\lambda|} \left(\frac{c^2}{|\lambda|} \right)^n < \beta \left(\frac{c^2}{|\lambda|} \right)^n.$$

Since, $c^2 < |\lambda|$ by the choice of c, we see that the difference of the subsequent functions converges uniformly and geometrically. From our initial choice of z_0 which was arbitrary inside the disk, we can say that $\{\phi_n\}$ converges uniformly on $\mathbb{D}_r$.

□

Corollary 5.4 (Global Linearisation)**.** *Given p, a geometrically attracting fixed point, there exists a holomorphic map ϕ from $\mathcal{A}$ to $\mathbb{C}$, where $\mathcal{A}$ is the basin of attraction of the fixed point, such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{C} & \xrightarrow{\lambda\cdot} & \mathbb{C} \end{array}$$

ϕ takes a neighbourhood of p biholomorphically onto a neighbourhood of 0 in $\mathbb{C}$. Further, this ϕ is unique up to multiplication by a constant.

Proof. The construction of such a ϕ is done simply by analytically extending the ϕ in Theorem 5.3 to the basin of attraction $\mathcal{A}$. Let U be the neighbourhood as in Theorem 5.3.

Take an arbitrary $z \in \mathcal{A}$. By the definition of $\mathcal{A}$, we know that the forward orbit of this point converges to p. So for some large enough n, $f^{\circ n}(z)$ falls inside U. This U can be holomorphically identified with $\mathbb{D}_\epsilon$ by ϕ. Here we can take the inverse with respect to the $\lambda\cdot$ fucntion n times, and define that to be the image of z under

ϕ. So we simply define $\phi(z)$ as

$$\phi(z) = \frac{\phi(f^{\circ n}(z))}{\lambda^n}.$$

We can also see this in the commutative diagram setup. We know that $\mathcal{A} = \bigcup f^{\circ -n}(U)$. The domain of ϕ can be extended to $f^{-1}(U)$ by making it the map that makes the diagram commute.

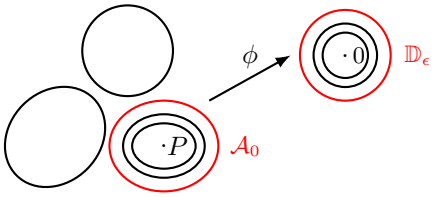
$$\begin{array}{ccccc} f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{f} & f(U) \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{D}_\delta & \underset{\frac{1}{\lambda}\cdot}{\overset{\lambda\cdot}{\rightleftarrows}} & \mathbb{D}_\epsilon & \xrightarrow{\lambda\cdot} & \mathbb{D}_\epsilon \end{array}$$

$\tilde{\phi}$ continually applying this method of extension, we get,

$$\begin{array}{ccccccc} A & \cdots & f^{-1}(U) & \xrightarrow{f} & U & \xrightarrow{f} & f(U) \\ \downarrow{\scriptstyle \phi} & \cdots & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ C & \cdots & \mathbb{D}_\delta & \xrightarrow{\lambda\cdot} & \mathbb{D}_\epsilon & \xrightarrow{\lambda\cdot} & \mathbb{D}_\epsilon \end{array}$$

□

Remark 5.5. Take f to be a rational function of degree ≥ 2, $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Let 0 be a geometrically attracting fixed point, and $\mathcal{A}_0$ its immediate basin of attraction. In the above theorem, when we extended the local linearisation function ϕ to the basin of attraction, note that instead of taking the entire inverse image $f^{-1}(U)$, we can extend it the connected component of $f^{-1}(U)$ that contains the fixed point. This is done to preserve the injectivity of the map. On this domain, ϕ will always remain biholomorphic and thus, invertible.



Continually extending ϕ to the connected component of $f^{-1}(U)$ that contains the fixed point, we will eventually get a map $\phi : \mathcal{A}_0 \to \mathbb{D}_\epsilon$. This ϕ will also be invertible on all of $\mathbb{D}_\epsilon$. Similarly to the above proof, the inverse at every step is extended to

make the following diagram commute with ψ.

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f} & U \\ \psi \uparrow \downarrow \phi & & \psi \uparrow \downarrow \phi \\ \mathbb{D}_\delta & \xrightarrow{\lambda \cdot} & \mathbb{D}_\epsilon \end{array}$$

Lemma 5.6 (Finding a critical point)**.** *Let the setup be same as Remark 5.5*

1. *The local inverse, $\psi_\epsilon : \mathbb{D}_\epsilon \to \mathcal{A}_0$ can be extended by analytic continuation to $\psi_r : \mathbb{D}_r \to A_0$, for some maximal disk $\mathbb{D}_r$. This ψ_r is a unique and holomorphic map, with $\psi(0) = p$, the fixed point, and $\phi \circ \psi(w) = w$ for all $w \in \mathbb{D}_r$.*
2. *Further, ψ can be homeomorphically extended over the boundary of the disk, $\partial\mathbb{D}_r$, and the image of this boundary in $\mathcal{A}_0$ necessarily has a critical point of f on it.*

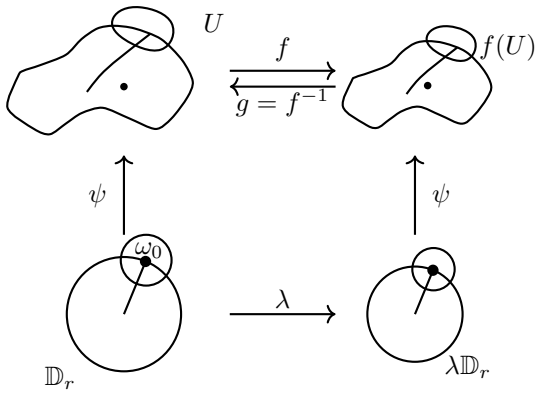
Proof.

(1) We want to show that there is a maximal disk $\mathbb{D}_r$ that ψ_ϵ can be extended to. For sake of contradiction, assume that we extend ψ_ϵ indefinitely, to all of $\mathbb{C}$. So the image of the complex plane is an open set in $\mathcal{A}_0$. That is, $\psi(\mathbb{C}) \subset \mathcal{A}_0 \subset \hat{\mathbb{C}}$. By Picard's theorem, we can say that since ψ is an entire holomorphic function, it misses almost one value. So, $\hat{\mathbb{C}} \setminus \psi(\mathbb{C})$ has almost one point. Since, f is injective on $\psi(\mathbb{C})$, it is injective on all of $\hat{\mathbb{C}}$. This contradicts the fact that f has degree greater than 2.
So our assumption must be false, and we must only be able to extend ψ_ϵ to a maximal disk $\mathbb{D}_r$. Let $U = \psi(\mathbb{D}_r) \subset A_0$. The following diagram of biholomorphisms commutes.

$$\begin{array}{ccc} U & \xrightarrow{f} & f(U) \\ \psi \uparrow \downarrow \phi & & \psi \uparrow \downarrow \phi \\ \mathbb{D}_r & \xrightarrow{\lambda \cdot} & \lambda\mathbb{D}_r \end{array}$$

(2) Now, to show that the boundary $\partial\mathbb{D}_r$ contains a critical point. We argue that if there is no critical point on the boundary, then ψ can be analytically extended to the boundary. Let w_0 be any point on ∂D_r. Consider the curve $\gamma : t \mapsto \psi(tw_0)$, where t takes values < 1. Since ψ is defined on $\mathbb{D}_r$ and $tw_0 \in \mathbb{D}_r$ γ is a curve on U. Let z_0 be a limit point of this curve as $t \to 1$. This z_0 is a candidate for $\psi(w_0)$.
If z_0 is not a critical point, then f is biholomorphic on some neighbourhood of $\mathcal{N}(z_0)$. So f^{-1} is defined on some neighbourhood $\mathcal{N}(f(z_0))$. We also

know ϕ is biholomorphic, so we can find a neighbourhood of a point on $\partial\lambda\mathbb{D}_r$ that is sent to $\mathcal{N}(f(z_0))$ under ψ. And by the inverse of the $\lambda\cdot$, a neighbourhood of w_0 which is sent to this.



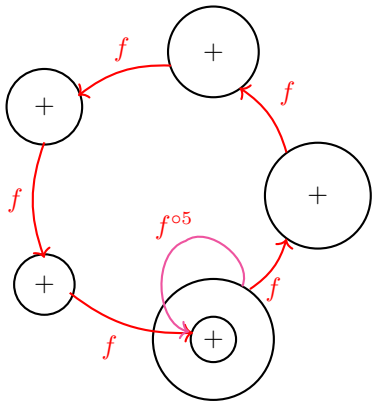
We can extend ψ to this neighbourhood of w_0 by defining it as

$$\psi : w \mapsto g \circ \psi(\lambda w)$$

If w_0 is varied all over $\partial\mathbb{D}_r$, and we don't encounter a critical point z_0 then there is a neighbourhood around all of those points that ψ can be extended to. So ψ can be extended to a bigger disk than $\mathbb{D}_r$. But since $\mathbb{D}_r$ was maximal that is not possible. So we must have a critical point on ∂U.

□

Now we talk about attracting periodic orbits. let the attracting orbit be $z_1 \to z_2 \to \cdots \to z_m \to z_1$. All of these orbit points are fixed points of the iterate $f^{\circ m}$. The immediate basin of attraction of this periodic orbit is simply the union of the immediate basins of attraction of all the points in the orbit. Pictorially, this is the behaviour around an attracting periodic orbit. This is an example of the case of $f^{\circ 5}$.



Theorem 5.7. *If f is a rational function of degree ≥ 2, then the immediate basin of attraction every attracting periodic orbit of f contains atleast one critical point of f. So the number of attracting periodic orbits of f is at most the number of critical points of f.*

Proof. First, consider the case of the orbit being geometrically attracting, which is that each point is a geometrically attracting fixed point of $f^{\circ m}$. Using Lemma 5.6 on any of these points, we know that there exists a maximal disk $\mathbb{D}_r$ around the point, such that the boundary of this disk contains a critical point of $f^{\circ m}$.

Consider the attracting periodic orbit $z_0 \to z_1 \to z_2 \to \cdots \to z_{m-1} \to z_0$ of the point z_0 under f. We know f maps $\mathcal{A}_0(z_j)$ into $\mathcal{A}_0(z_{j+1})$ for all j. So, $f^{\circ m}(\mathcal{A}_0(z_j)) \subset \mathcal{A}_0(z_j)$ for all j. If no $\mathcal{A}_0(z_j)$ contained a critical point of f, then $f'(z)$ is non-zero for all $z \in \mathcal{A}$. And we also know that f maps the basin to itself. By the chain rule, we have

$$(f^{\circ m})'(z) = f'(f^{\circ m-1}(z)) \cdot f'(f^{\circ m-2}(z)) \cdots f'(z)$$

Since $f'(z)$ is non-zero for all $z \in \mathcal{A}$, we can say that $(f^{\circ m})'(z)$ is non-zero for all $z \in \mathcal{A}$. This contradicts the fact that $f^{\circ m}$ has a critical point in the basin of attraction of the periodic orbit. So we must have a critical point in the basin of attraction of the periodic orbit.

The second part of the theorem follows from the fact that because f is holomorphic, it is rational, and thus has finitely many critical points. So the number of attracting periodic orbits is at most the number of critical points of f. □

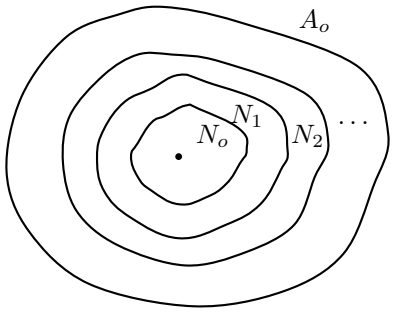
By the Riemann-Hurwitz (2.9), we know that if f is meromorphic, it has at most $2d-2$ critical points, where d is the degree of f. So the number of attracting periodic orbits is at most $2d-2$.

Lemma 5.8 (Topology of $\mathcal{A}_0$)**.** *Let $\mathcal{A}_0$ be the immediate basin of an attracting fixed point p of $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Then the complement $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ is either connected or has uncountably many connected components.*

Proof. Consider N_0, a small neighbourhood of p, such that $f(\bar{N_0}) \subset N_0$, and the boundary of N_0 is a simple closed curve γ. Let this be such that γ contains no forward image of a critical point of f. Define N_k to be the connected component of $f^{-k}(N_0)$ that contains p So $N_k = f^{-k}(N_0) \cap \mathcal{A}_0$. Also from the definition of the basin of attraction, we know that $\mathcal{A} = \bigcup_{k=0}^{\infty} f^{\circ -k}(N_0)$. So,

$$\bigcup_{k=0}^{\infty} N_k = \bigcup_{k=0}^{\infty} (f^{\circ -k}(N_0) \cap \mathcal{A}_0) = \mathcal{A}_0 \cap \bigcup_{k=0}^{\infty} f^{\circ -k}(N_0) = \mathcal{A}_0 \cap \mathcal{A} = \mathcal{A}_0$$

Since each N_k is a connected domain we know that $\partial \mathbb{N}_k$ is the union of finitely many simple closed curves, each of which is the boundary of a connected component of $f^{-k}(N_0)$. So, ∂N_k is a finite union of simple closed curves. And $\hat{\mathbb{C}} \setminus N_k$ is the union of the same number of topological disks.

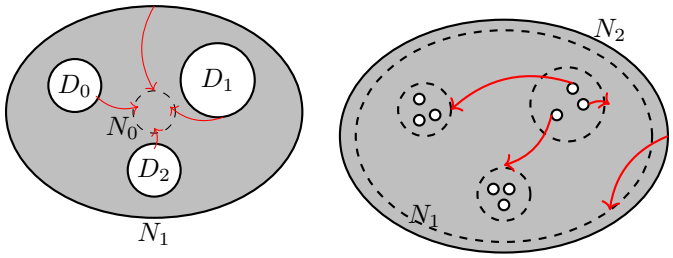


Case 1: If each N_k is bound by a single simple closed curve, as in the figure above, then $\hat{\mathbb{C}} \setminus N_k$ is connected and thus $\hat{\mathbb{C}} \setminus \mathcal{A}_0$ is connected.

Case 2: If each N_k is bound by multiple simple closed curves, then we argue as follows. We know each N_k is bound by a finite number of simple closed curves. WLOG, let N_0 be bound by a single simple closed curve and N_1 be bound by more than one simple closed curve. This can be done since if you start with any N_0 and Case 1 doesn't hold, then N_k has multiple simple closed curves as boundaries. Simply take N_{k-1} to be N_0.

Rather than a complete argument, we give a sketch of it. Let N_1 be bound by $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. We know that for all k, f maps ∂N_k to ∂N_{k-1}, and thus every boundary component of N_k is mapped to a boundary component of N_{k-1}. One can further state, that for every boundary component Γ_i of N_k, there exists a boundary component Γ_j of N_{k-1} such that $f(\Gamma_i) = \Gamma_j$. This can be shown inductively, but is left out here. The reader can find it in Milnor's book [1].

So this tells us for every sequence $(i_1, i_2, \ldots, i_k)$ of integers from 1 to n there exists a boundary component of N_k such that $f(\Gamma_{i_1 i_2 \ldots i_k}) = \Gamma_{i_2 i_3 \ldots i_k}$. So pictorially, for any sequence, the boundary component of N_k jumps around to components of N_{k-1}, contained inside any disk D_{i_2} of $\hat{\mathbb{C}} \setminus N_1$. Refer to the figure below for a pictorial representation of this.



□

Definition 5.9 (Topologically Repelling Fixed Point)**.** A fixed point p of f is said to be topologically repelling if there exists a neighbourhood N of p such that every point in N except p is mapped outside of N by some iterate of f. Such an N is called a *forward isolating neighbourhood* of p.

Lemma 5.10. *A point is topologically repelling if and only if the multiplier of f at that point, $\lambda > 1$.*

Proof.

$\Leftarrow$ If $\lambda > 1$, it follows from the Koenigs linearisation Theorem (5.3) that there exists a neighbourhood N of p such that for all $z \in N$, and a map ϕ such that $\phi \circ f(z) = \lambda \cdot \phi(z)$.
Choose N to be compact forward isolating neighbourhood of p. Then, $f(N)$ is also a compact neighbourhood of p. Define N_k to be the compact neighbourhood of points for which the first k^{th} iterates lie inside N. That is,

$$N_k = N \cap f^{-1}(N) \cap f^{-2}(N) \cap \cdots \cap f^{-k}(N)$$

We can see that $N_k = N_{k-1} \cap f^{-k}(N)$. So $N \supset N_1 \supset N_2 \supset \cdots \supset \ldots$
Also,

$$\begin{aligned} N_k &= \bigcap_{j=1}^{k} f^{-j}(N) \\ f(N_k) &= \bigcap_{j=0}^{k} f(f^{-j}(N)) \\ &= f(N) \cap N \cap N_1 \cap N_2 \cap \cdots \cap N_{k-1} \\ &= F(N) \cap N_{k-1} \end{aligned}$$

where $N_{k-1} \subset f(N_k)$ for k large, since the diameters tend to zero. Thus

$$f(N_k) = N_{k-1} \quad \text{for sufficiently large } k,$$

and in fact f maps N_k homeomorphically onto N_{k-1}. Now let U_k be the connected component of $\text{int}(N_k)$ containing p. Then f maps U_k biholomorphically onto the strictly smaller set U_{k-1}. By the Schwarz Lemma its multiplier λ satisfies $|\lambda| < 1$, which forces $|f'(p)| > 1$ as required.

$\Rightarrow$ To show that if the multiplier is greater than 1, then the point is topologically repelling, we use the following argument. Let f be a holomorphic function with 0 as a fixed point with multiplier greater than 1. By Taylor's theorem, on some neighbourhood of 0, we can write

$$\begin{aligned} & f(z) = \lambda z + o(z^2) && \text{as } z \to 0. \\ \implies & |f(z) - \lambda z| < \alpha |z^2| && \text{for some constant } \alpha > 0, \\ \implies & |f(z)| > |\lambda z| - \alpha |z^2| \end{aligned}$$

Our goal is to show that f strictly increases distance from 0 near the origin; that is, there exists a neighbourhood $|z| < \epsilon$ such that $|f(z)| > |z|$ for all $z \neq 0$.

Let $|z| < \epsilon$ for some small $\epsilon > 0$. Then we have

$$|f(z)| > |\lambda||z| - \alpha|z^2| = (|\lambda| - \alpha\epsilon)|z|$$

Here, we can choose ϵ small enough such that $\lambda - \alpha\epsilon > 1$. This gives us

$$|f(z)| > |z| \qquad \forall z \in \mathbb{D}_\epsilon$$

So, we can take $N = \mathbb{D}_\epsilon$ as the forward isolating neighbourhood of 0. This shows that 0 is a topologically repelling fixed point of f.

□

Remark 5.11. The Koenigs linearisation theorem (5.3) and the fact that repelling periodic orbits lie in the Julia sets are useful to get a heuristic idea of why the

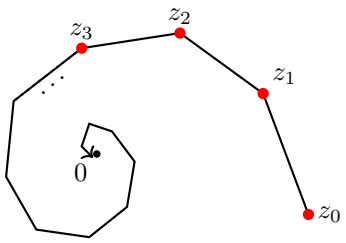
Julia set is a fractal. Consider a neighbourhood of a repelling periodic point. By Theorem 5.3, we know that the local action of f on this neighbourhood is complex scaling. So this neighbourhood is mapped to a neighbourhood which has a very similar structure, but might be scaled and rotated. This gives rise to the self-similar structure of the Julia set.

Corollary 5.12. *Let p be a repelling periodic point of f, a meromorphic function. Let p be such that the multiplier λ at p is a non-real complex number. Then the Julia set $\mathcal{J}(f)$ cannot be a smooth manifold, unless it is all of $\hat{\mathbb{C}}$.*

We start with the case that the Julia set is not all of $\hat{\mathbb{C}}$. Take z_0, a point close to p in the Julia set. Look at this point in the local Koenigs coordinates, take the backward orbit using the inverse of the $\lambda\cdot$ map. So let $w_k = \phi(z_0)$. And $z_0 \leftarrow z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_k$ be such that

$$\phi(z_k) = \lambda^{-k} w_0$$

Since λ is a non-real complex number, multiplication by λ^{-k} will rotate the point w_k in the complex plane. Also since $|\lambda| > 1$, the distance from the origin will decrease as k increases. So the points z_k will spiral around the origin. Such a spiral cannot be a smooth manifold, since it has a cusp at the origin. So the Julia set cannot be a smooth manifold, unless it is all of $\hat{\mathbb{C}}$.



Lemma 5.13 (Global extension of ϕ^{-1})**.** *Given f a holomoprhic map from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$, and p a repelling fixed point of f, then there exists a holomorphic map $\psi : \mathbb{C} \to \hat{\mathbb{C}}$ such that $\psi(0) = p$ and $\phi \circ \psi(w) = w$ for all $w \in \mathbb{C}$. Here, ϕ is the Koenigs local linearisation of f at p from Theorem 5.3.*

In the commutative diagram setup, the following diagram commutes

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f} & \hat{\mathbb{C}} \\ \uparrow{\scriptstyle \psi} & & \uparrow{\scriptstyle \psi} \\ \mathbb{C} & \xrightarrow{\lambda\cdot} & \mathbb{C} \end{array}$$

Proof. We start with $\psi_\epsilon : \mathbb{D}_\epsilon \to \hat{\mathbb{C}}$, the branch of the inverse of ϕ that maps 0 to p. We extend the domain of this ψ_ϵ to an arbitrary large disk $\mathbb{D}_r$ by defining it as

$$\psi(w) = f^{\circ n}\left(\psi_\epsilon\left(\frac{w}{\lambda^n}\right)\right)$$

for all $w \in \mathbb{D}_r$. Since $|\lambda| > 1$, we can choose n large enough such that $\frac{w}{\lambda^n} \in \mathbb{D}_\epsilon$. This expression we get, can also be seen from the functional equation of the commutative diagram

$$\begin{array}{ccc} \hat{\mathbb{C}} & \xrightarrow{f^{\circ n}} & \hat{\mathbb{C}} \\ \uparrow{\scriptstyle \psi_\epsilon} & & \uparrow{\scriptstyle \psi} \\ \mathbb{D}_\epsilon & \xrightarrow{\lambda^n \cdot} & \mathbb{D}_r \end{array}$$

□

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